PART A

Unit - I

Numerical Methods - 1

1.1 Numerical Solution of Ordinary Differential Equations of First Order and First Degree - Introduction

We have studied various analytical (*theoretical*) methods of solving differential equations, applicable only to equations in some specific form. But the differential equations arising out of many physical problems do not belong to a specific form and some times analytical solution may not even exist. In some cases it may be very difficult to solve by analytical methods. In such cases **Numerical Methods** assumes importance and computers help in many numerical methods for obtaining the result to the highest degree of accuracy.

1.2 Numerical methods for initial value problems

Consider a differential equation of first order and first degree in the form $\frac{dy}{dx} = f(x, y)$ with the initial condition $y(x_0) = y_0$, that is $y = y_0$ at $x = x_0$.

This problem of finding y is called an ***initial value problem.***

We discuss several numerical methods for solving an initial value problem.

1.21 Picard's Method

This method gives the solution of the initial value problem in the form of a power series in x from which we can find the value of y at any number of values of x in the neighbourhood of the initial value x_0. The method is as follows.

Consider the first order differential equation :

$$\frac{dy}{dx} = f(x, y) \text{ with the initial condition } y(x_0) = y_0$$

That is, $dy = f(x, y)\, dx \;;\quad y = y_0$ at $x = x_0$

Integrating LHS between y_0 and y, RHS between x_0 and x we have,

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\, dx$$

i.e., $$y\Big]_{y_0}^{y} = \int_{x_0}^{x} f(x,y)\, dx \quad \text{or} \quad y - y_0 = \int_{x_0}^{x} f(x,y)\, dx$$

$$\therefore \qquad y = y_0 + \int_{x_0}^{x} f(x,y)\, dx \qquad \ldots (1)$$

The first approximation y_1 is obtained by putting $y = y_0$ in $f(x,y)$ and integrating the same. That is,

$$y_1 = y_0 + \int_{x_0}^{x} f(x,y_0)\, dx$$

The second approximation y_2 is obtained by putting $y = y_1$ in $f(x,y)$ of (1) and integrating the same. That is

$$y_2 = y_0 + \int_{x_0}^{x} f(x,y_1)\, dx$$

Continuing like this we get a series of approximations $y_1, y_2, y_3, \ldots$ of the initial value problem each being a better approximation than the preceding one. This method is referred to as the ***Picard's method of successive approximations*** which is carried out till we reach the desired/specified degree of accuracy.

This method can be extended to simultaneous and higher order differential equations.

WORKED PROBLEMS

1. *Use the Picard's method to obtain the fourth approximation to the solution of* $\frac{dy}{dx} = x + y,\ y(0) = 1$ *and hence find y at* $x = 0.1, 0.2$.

Solution : By data, $\frac{dy}{dx} = x + y,\ y(0) = 1$

i.e., $dy = (x + y)\, dx \ ; \quad y = 1$ at $x = 0$

Integtating LHS between 1 and y, RHS between 0 and x we have

$$\int_1^y dy = \int_0^x (x+y)\,dx \quad \text{or} \quad [y]_1^y = \int_0^x (x+y)\,dx$$

i.e., $$y - 1 = \int_0^x (x+y)\,dx$$

$$\therefore \quad y = 1 + \int_0^x (x+y)\,dx \qquad \ldots (1)$$

The first approximation (y_1) to the solution is obtained by putting $y = 1$ in the RHS of (1).

$$y_1 = 1 + \int_0^x (x+1)\,dx = 1 + \left[\frac{x^2}{2} + x\right]_0^x = 1 + x + \frac{x^2}{2}$$

The second approximation (y_2) is obtained by replacing y by y_1 in the RHS of (1).

$$y_2 = 1 + \int_0^x (x+y_1)\,dx$$

$$= 1 + \int_0^x \left[x + \left(1 + x + \frac{x^2}{2}\right)\right] dx$$

$$= 1 + \int_0^x \left(1 + 2x + \frac{x^2}{2}\right) dx$$

$$\therefore \quad y_2 = 1 + x + x^2 + \frac{x^3}{6}$$

$$y_3 = 1 + \int_0^x (x + y_2)\,dx$$

$$= 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{6}\right) dx$$

$$\therefore \quad y_3 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

$$y_4 = 1 + \int_0^x (x + y_3)\,dx$$

$$= 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right) dx$$

Thus $\quad \mathbf{y_4 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}}$

This is the fourth approximation to the solution of the given initial value problem. By putting $x = 0.1$ and $x = 0.2$ in the fourth approximation we have

$$y\,(0.1) = 1 + (0.1) + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{12} + \frac{(0.1)^5}{120} = 1.1103$$

$$y\,(0.2) = 1 + (0.2) + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{12} + \frac{(0.2)^5}{120} = 1.2428$$

Thus $\quad \mathbf{y(0.1) = 1.1103}$ **and** $\mathbf{y(0.2) = 1.2428}$

Note : (1) *The limits of integration 0 and x has no effect when the terms involved are in powers of x like for example : x^2 between 0 and x yields $x^2 - 0^2 = x^2$. However in the case of terms like e^x, $\cos x$ etc it does make a difference as we observe*

$$[e^x]_0^x = e^x - e^0 = e^x - 1 \;;\; [\cos x]_0^x = \cos x - \cos 0 = \cos x - 1 .$$

(2) ***We shall find the analytical solution*** *of the initial value problem as it helps us to correlate between the numerical solution and the analytical solution.*

The given equation can be written in the form $\frac{dy}{dx} - y = x$ which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$ whose exact solution is given by :

$$y\,e^{\int P dx} = \int Q e^{\int P dx}\,dx + c$$

Here $\quad P = -1,\ Q = x$. Hence we have $e^{\int P dx} = e^{-x}$

The solution is $\; y e^{-x} = \int x e^{-x}\,dx + c$

Integrating RHS by parts, we obtain,

$$y e^{-x} = x(-e^{-x}) - (1)\,(e^{-x}) + c$$

or $\quad y e^{-x} = -x e^{-x} - e^{-x} + c$

Multiplying by e^x we have $y = -x - 1 + c\,e^x$.

Using the initial condition $y = 1$ at $x = 0$ in the solution we have

$$1 = -0 - 1 + c \qquad \therefore \quad c = 2.$$

Substituting this value of c, the exact (*analytical or theoretical*) solution is given by

$$y = 2e^x - x - 1$$

Now using the series expansion of $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots$ the solution becomes,

$$y = 2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots\right) - x - 1$$

or $$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \cdots$$

Comparing this with our fourth approximation,

$$y_4 = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

we observe that y_4 approximates to the exact solution upto the term in x^4.

Also by putting $x = 0.1$ and 0.2 in the exact solution $y = 2e^x - x - 1$ we obtain

$$y(0.1) = 2e^{0.1} - 0.1 - 1 = \mathbf{1.1103418}$$

$$y(0.2) = 2e^{0.2} - 0.2 - 1 = \mathbf{1.2428055}$$

These theoretical values agrees with the values obtained from the fourth approximation, in the numerical method.

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2. *Use Picard's method to obtain the third approximation to the solution of* $\frac{dy}{dx} + y = e^x$, $y(0) = 1$ *and hence find* $y(0.2)$. *Find also the true value by solving the given equation.*

Solution : By data, $\frac{dy}{dx} + y = e^x$ or $\frac{dy}{dx} = e^x - y$

$$dy = (e^x - y)\,dx \quad ; \quad y = 1,\ x = 0$$

Integrating LHS between 1 and y, RHS between 0 and x, we have,

$$\int_1^y dy = \int_0^x (e^x - y)\,dx \quad \text{or} \quad y - 1 = \int_0^x (e^x - y)\,dx$$

$$\therefore \quad y = 1 + \int_0^x (e^x - y)\,dx \qquad \ldots(1)$$

The first approximation (y_1) to the solution is obtained by putting $y = 1$ in the RHS of (1).

$$y_1 = 1 + \int_0^x (e^x - 1)\, dx$$

$$= 1 + [e^x - x]_0^x = 1 + (e^x - x) - (e^0 - 0)$$

$$= 1 + e^x - x - 1$$

$$\therefore \quad y_1 = e^x - x$$

$$y_2 = 1 + \int_0^x (e^x - y_1)\, dx$$

$$= 1 + \int_0^x [e^x - (e^x - x)]\, dx = 1 + \int_0^x x\, dx$$

$$\therefore \quad y_2 = 1 + \left[\frac{x^2}{2}\right]_0^x = 1 + \frac{x^2}{2}$$

$$y_3 = 1 + \int_0^x (e^x - y_2)\, dx$$

$$= 1 + \int_0^x \left[e^x - \left(1 + \frac{x^2}{2}\right)\right] dx$$

$$= 1 + \left[e^x - x - \frac{x^3}{6}\right]_0^x = 1 + \left(e^x - x - \frac{x^3}{6}\right) - 1$$

Thus $y_3 = e^x - x - \dfrac{x^3}{6}$ is the required third approximation.

Putting $x = 0.2$ we have $y(0.2) = e^{0.2} - 0.2 - \dfrac{(0.2)^3}{6} = \mathbf{1.0200694}$.

We shall now find the analytical solution of $\dfrac{dy}{dx} + y = e^x$ which is of the form $\dfrac{dy}{dx} + Py = Q$ whose solution is given by

$$y\, e^{\int P dx} = \int Q e^{\int P dx}\, dx + c$$

Here $P = 1$, $Q = e^x$; $e^{\int P dx} = e^x$

Hence we have $y e^x = \int e^x \cdot e^x \, dx + c$

i.e., $y e^x = \int e^{2x} \, dx + c$ or $y e^x = \frac{e^{2x}}{2} + c$

Multiplying by e^{-x}, $y = \frac{e^x}{2} + c e^{-x}$

Now substituting the initial condition $y = 1$, $x = 0$ we have, $1 = 1/2 + c$ or $c = 1/2$.

The solution now becomes

$$y = \frac{e^x}{2} + \frac{1}{2} e^{-x} = \frac{1}{2} (e^x + e^{-x}) = \cos hx$$

Putting $x = 0.2$, $y = \frac{1}{2} (e^{0.2} + e^{-0.2}) = \mathbf{1.0200668}$

It may be observed that the numerical value and the theoretical value coincide upto the fifth decimal place.

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3. *Employ Picard's method to obtain the solution of the differential equation* $\frac{dy}{dx} = x^2 + y^2$ *, given that* $y = 0$ *when* $x = 0$ *. Hence find* $y(0.1)$ *correct to four places of decimal .*

\>> Since the number of approximations is not specified we shall find two approximations to the solution.

By data $\frac{dy}{dx} = x^2 + y^2$, $y = 0$ $x = 0$

$$dy = (x^2 + y^2) \, dx$$

Integrating LHS between 0 and y, RHS between 0 and x we have,

$$\int_0^y dy = \int_0^x (x^2 + y^2) \, dx$$

$$\therefore \quad y = \int_0^x (x^2 + y^2) \, dx \qquad \ldots (1)$$

Putting $y = 0$ in the RHS of (1), $y_1 = \int_0^x x^2 \, dx = \frac{x^3}{3}$

$$y_2 = \int_0^x (x^2 + y_1^2)\, dx = \int_0^x [x^2 + (x^3/3)^2]\, dx$$

$$= \int_0^x \left(x^2 + \frac{x^6}{9}\right) dx = \frac{x^3}{3} + \frac{x^7}{63}$$

Thus the second approximation is $y = \dfrac{x^3}{3} + \dfrac{x^7}{63}$

Now $y\,(0.1) = \dfrac{(0.1)^3}{3} + \dfrac{(0.1)^7}{63} = 0.00033$

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4. *Use Picard's method to obtain the second approximation to the solution of* $\dfrac{dy}{dx} = x - y^2$ *,* $x_0 = 0 \;\; y_0 = 1$ *initially. Hence compute* $y\,(0.1)$ *correct to four decimal places .*

>> By data $\dfrac{dy}{dx} = x - y^2$; $x_0 = 0,\ y_0 = 1$

$$dy = (x - y^2)\, dx \quad ; \quad y = 1,\ x = 0 \text{ initially.}$$

Integrating LHS between 1 and y , RHS between 0 and x we have,

$$\int_1^y dy = \int_0^x (x - y^2)\, dx \quad \text{or} \quad y - 1 = \int_0^x (x - y^2)\, dx$$

$$\therefore \quad y = 1 + \int_0^x (x - y^2)\, dx.$$

$$y_1 = 1 + \int_0^x (x - 1^2)\, dx = 1 + \left[\frac{x^2}{2} - x\right]_0^x = 1 - x + \frac{x^2}{2}$$

$$y_2 = 1 + \int_0^x (x - y_1^2)\, dx$$

$$= 1 + \int_0^x \left[x - \left(1 - x + \frac{x^2}{2}\right)^2\right] dx$$

$$= 1 + \int_0^x \left[x - \left(1 + x^2 + \frac{x^4}{4} - 2x - x^3 + x^2\right)\right] dx$$

where we have used $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$$y_2 = 1 + \int_0^x \left(-1 + 3x - 2x^2 + x^3 - \frac{x^4}{4} \right) dx$$

Thus $y_2 = 1 - x + \frac{3x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} - \frac{x^5}{20}$ is the required second approximation.

Putting $x = 0.1$ in this expression we obtain $y(0.1) = 0.9144$.

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5. *Use Picard's method to find* y *at* $x = 0.25$, 0.5, 0.75 *given* $\frac{dy}{dx} = \frac{x^2}{y^2 + 1}$, $y(0) = 1$ *by taking two approximations.*

\>> By data $\frac{dy}{dx} = \frac{x^2}{y^2 + 1}$; $y = 1$, $x = 0$

$$dy = \left[\frac{x^2}{y^2 + 1} \right] dx$$

Integrating LHS between 1 and y, RHS between 0 and x we have,

$$\int_1^y dy = \int_0^x \frac{x^2}{y^2 + 1} dx \quad \text{or} \quad y - 1 = \int_0^x \frac{x^2}{y^2 + 1} dx$$

$$\therefore \quad y = 1 + \int_0^x \frac{x^2}{y^2 + 1} dx \qquad \dots (1)$$

Putting $y = 1$ in the RHS of (1),

$$y_1 = 1 + \int_0^x \frac{x^2}{1^2 + 1} dx = 1 + \left[\frac{x^3}{6} \right]_0^x = 1 + \frac{x^3}{6}$$

$$y_2 = 1 + \int_0^x \frac{x^2}{y_1^2 + 1} dx = 1 + \int_0^x \frac{x^2}{\left(1 + \frac{x^3}{6} \right)^2 + 1} dx$$

Putting $1 + \frac{x^3}{6} = t$, we have $\frac{3x^2}{6} dx = dt$ or $x^2\, dx = 2\, dt$

Hence, $y_2 = 1 + \int \frac{2\, dt}{t^2 + 1} = 1 + 2 \tan^{-1} t$

i.e., $y_2 = 1 + [\,2\tan^{-1}(1+x^3/6)\,]_0^x$

$y_2 = 1 + 2\tan^{-1}(1+x^3/6) - 2\tan^{-1}(1)$. But $\tan^{-1}(1) = \pi/4$

Thus $\mathbf{y_2 = 1 - (\pi/2) + 2\tan^{-1}(1+x^3/6)}$

We shall now find y at $x = 0.25, 0.5, 0.75$ from the second approximation taking the value of π in radian measure.

$$y(0.25) = 1 - \pi/2 + 2\tan^{-1}\left[1 + \frac{(0.25)^3}{6}\right] = 1.0026$$

Similarly we have to compute $y(0.5)$ and (0.75).

Thus $\mathbf{y(0.25) = 1.0026,\ y(0.5) = 1.0206,\ y(0.75) = 1.0679}$

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6. *Use Picard's method to solve* $\dfrac{dy}{dx} = x + y,\ y(0) = 1$ *in the range* $0 \le x \le 0.2$ *with step size* $h = 0.1$. *Perform two iterations in each step.*

>> The problem has to be done in two stages since the step size is specified as 0.1 We must find two approximations using the initial condition $y(0) = 1$ and compute $y(0.1)$. This answer must be used as the initial condition to again find two approximations and compute $y(0.2)$ from it.

I Stage : $dy = (x+y)\,dx;\ y = 1,\ x = 0$

Integrating L.H.S between 1 and y, RHS between 0 and x we have,

$$\int_1^y dy = \int_0^x (x+y)\,dx \quad \text{or} \quad y - 1 = \int_0^x (x+y)\,dx$$

$$\therefore \quad y = 1 + \int_0^x (x+y)\,dx \qquad \dots (1)$$

From (1), $y_1 = 1 + \displaystyle\int_0^x (x+1)\,dx = 1 + \left[x + \frac{x^2}{2}\right]_0^x$

$$\therefore \quad y_1 = 1 + x + \frac{x^2}{2}$$

Again from (1),

$$y_2 = 1 + \int_0^x (x + y_1)\,dx = 1 + \int_0^x \left(1 + 2x + \frac{x^2}{2}\right) dx$$

$\therefore \quad y_2 = 1 + x + x^2 + \dfrac{x^3}{6}$

Now $\quad y(0.1) = 1 + (0.1) + (0.1)^2 + \dfrac{(0.1)^3}{6} = 1.1102 \approx 1.11$

II Stage : $dy = (x + y)\,dx \;;\; y = 1.11,\; x = 0.1$

Integrating LHS between 1.11 and y, RHS between 0.1 and x, we have,

$$\int_{1.11}^{y} dy = \int_{0.1}^{x} (x + y)\,dx \quad \text{or} \quad y - 1.11 = \int_{0.1}^{x} (x + y)\,dx$$

$$\therefore \quad y = 1.11 + \int_{0.1}^{x} (x + y)\,dx \qquad \ldots (2)$$

From (2), $y_1 = 1.11 + \displaystyle\int_{0.1}^{x} (x + 1.11)\,dx$

$$= 1.11 + \left[1.11\,x + \frac{x^2}{2}\right]_{0.1}^{x}$$

$$= 1.11 + \left(1.11\,x + \frac{x^2}{2}\right) - \left(1.11 \times 0.1 + \frac{(0.1)^2}{2}\right)$$

$$\therefore \quad y_1 = 0.994 + 1.11\,x + \frac{x^2}{2}$$

Again from (2) $\quad y_2 = 1.11 + \displaystyle\int_{0.1}^{x} (x + y_1)\,dx$

i.e., $\quad y_2 = 1.11 + \displaystyle\int_{0.1}^{x} \left[x + \left(0.994 + 1.11x + \frac{x^2}{2}\right)\right] dx$

$$= 1.11 + \int_{0.1}^{x} \left(0.994 + 2.11\,x + \frac{x^2}{2}\right) dx$$

$$= 1.11 + \left[0.994\,x + 2.11\frac{x^2}{2} + \frac{x^3}{6}\right]_{0.1}^{x}$$

$$= 1.11 + \left[0.994\,x + 1.055x^2 + \frac{x^3}{6}\right]_{0.1}^{x}$$

$$y_2 = 1.11 + \left(0.994\,x + 1.055\,x^2 + \frac{x^3}{6}\right) - \left(0.994 \times 0.1 + 1.055 \times 0.01 + \frac{0.001}{6}\right)$$

$$\therefore \quad y_2 = 0.9999 + 0.994\,x + 1.055\,x^2 + \frac{x^3}{6}$$

Now putting $x = 0.2$ in this expression, we obtain $y(0.2) = \mathbf{1.2422}$

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7. *Compute $y(0.1)$ and $y(0.2)$ by solving $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$ by Picard's method.*

>> By data, $dy = \left(\frac{y-x}{y+x}\right) dx$; $y = 1,\ x = 0.$

Integrating LHS between 1 and y, RHS between 0 and x we have,

We have, $\int_1^y dy = \int_0^x \left(\frac{y-x}{y+x}\right) dx$ or $y - 1 = \int_0^x \left(\frac{y-x}{y+x}\right) dx$

$$\therefore \quad y = 1 + \int_0^x \left(\frac{y-x}{y+x}\right) dx \qquad \ldots (1)$$

Putting $y = 1$ in the RHS of (1) we have,

$$y_1 = 1 + \int_0^x \left(\frac{1-x}{1+x}\right) dx$$

Let $\quad 1 - x = A(1+x) + B \quad$ or $\quad 1 - x = (A+B) + Ax$

That implies $1 = A + B$ and $-1 = A$. Hence $B = 2$

$\therefore \quad 1 - x = -1(1+x) + 2$

Now, $\quad y_1 = 1 + \int_0^x \left[\frac{-(1+x)+2}{1+x}\right] dx$

$$= 1 + \int_0^x \left[-1 + \frac{2}{1+x}\right] dx$$

$$= 1 + \Big[-x + 2\log(1+x)\Big]_0^x$$

$\therefore \quad \mathbf{y_1 = 1 - x + 2\log(1+x)}$, since $\log 1 = 0$.

From this expression we obtain $y(0.1) = \mathbf{1.0906}$ and $y(0.2) = \mathbf{1.1646}$

Remark : *We cannot find the second approximation.*

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8. *Compute y for x in the range 0 (0.25) 1 using Picard's second approximation in solving the differential equation* $(1+y^2)\ dy = x^2\ dx$ *given that* $y = 0$ *when* $x = 0$ *initially.*

>> It should be noted that the notation 0 (0.25) 1 means that the range $0 \le x \le 1$ needs to be covered at a step size of 0.25. We need to compute the value of y at $x = 0.25, 0.5, 0.75$ and 1 [$y(0) = 0$ *by data*] from the Picard's second approximation.

We have, $$dy = \left[\frac{x^2}{1+y^2}\right] dx \ ; \ y = 0, \ x = 0$$

Integrating LHS between 0 and y, RHS between 0 and x we have,

$$\int_0^y dy = \int_0^x \left[\frac{x^2}{1+y^2}\right] dx$$

$$\therefore \quad y = \int_0^x \left[\frac{x^2}{1+y^2}\right] dx \qquad \ldots(1)$$

Putting $y = 0$ in the RHS of (1) we have,

$$y_1 = \int_0^x x^2\ dx = \left[\frac{x^3}{3}\right]_0^x = \frac{x^3}{3}$$

$$y_2 = \int_0^x \left[\frac{x^2}{1+y_1^2}\right] dx = \int_0^x \left[\frac{x^2}{1+(x^3/3)^2}\right] dx$$

Putting $x^3/3 = t$, we have $x^2\ dx = dt$

$$y_2 = \int \frac{dt}{1+t^2} = \tan^{-1} t = \left[\tan^{-1}(x^3/3)\right]_0^x$$

Thus $\quad y_2 = \tan^{-1}(x^3/3)$

From this expression we compute y at $x = 0.25$, 0.5, 0.75 and 1.

Thus, $\mathbf{y(0.25) = 0.0052}$, $\mathbf{y(0.5) = 0.0416}$, $\mathbf{y(0.75) = 0.1397}$, $\mathbf{y(1) = 0.3218}$

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9. *Find the values of y for* $x = 0\,(0.1)\,0.5$ *given* $y' = 1 + xy$, *which passes through* (0 , 1) *from the Picard's third approximation.*

>> By data, $\dfrac{dy}{dx} = 1 + xy$ and $y(0) = 1$

ie., $\quad dy = (1 + xy)\ dx \ ; \ y = 1, x = 0$

Integrating LHS between 1 and y, RHS between 0 and x we have,

$$\int_1^y dy = \int_0^x (1+xy)\, dx \quad \text{or} \quad y-1 = \int_0^x (1+xy)\, dx$$

$$\therefore \quad y = 1+\int_0^x (1+xy)\, dx \qquad \ldots (1)$$

From (1), $y_1 = 1+\int_0^x (1+x)\, dx = 1+\left[x+\frac{x^2}{2} \right]_0^x$

$$\therefore \quad y_1 = 1+x+\frac{x^2}{2}$$

$$y_2 = 1+\int_0^x (1+xy_1)\, dx = 1+\int_0^x (1+x+x^2+\frac{x^3}{2})\, dx$$

$$\therefore \quad y_2 = 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}$$

$$y_3 = 1+\int_0^x (1+xy_2)\, dx = 1+\int_0^x (1+x+x^2+\frac{x^3}{2}+\frac{x^4}{3}+\frac{x^5}{8})\, dx$$

$$\therefore \quad y_3 = 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}+\frac{x^5}{15}+\frac{x^6}{48}$$

From this expression we obtain

$y(0.1) = 1.1053$ **$y(0.2) = 1.2229$** **$y(0.3) = 1.355$**

$y(0.4) = 1.5053$ **$y(0.5) = 1.6769$**

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10. *Solve* $\frac{dy}{dx}+y = 2x$ *by Picard's method given that* $y = 3$ *at* $x = 1$ *initially. Also compute* $y(1.1)$ *from the third approximation.*

>> By data, $\frac{dy}{dx} = 2x-y$

ie., $dy = (2x-y)\, dx\ ;\ y = 3,\ x = 1$

Integrating LHS between 3 and y, RHS between 1 and x we have,

$$\int_3^y dy = \int_1^x (2x-y)\, dx \quad \text{or} \quad y-3 = \int_1^x (2x-y)\, dx$$

$$\therefore \quad y = 3+\int_1^x (2x-y)\, dx \qquad \dots(1)$$

From (1), $y_1 = 3+\int_1^x (2x-3)\, dx$

ie., $y_1 = 3+[x^2-3x]_1^x = 3+(x^2-3x)-(1-3)$

$$\therefore \quad y_1 = 5-3x+x^2$$

$$y_2 = 3+\int_1^x (2x-y_1)\, dx$$

$$= 3+\int_1^x (-5+5x-x^2)\, dx$$

$$= 3+\left[-5x+\frac{5x^2}{2}-\frac{x^3}{3}\right]_1^x$$

$$= 3+\left(-5x+\frac{5x^2}{2}-\frac{x^3}{3}\right)-\left(-5+\frac{5}{2}-\frac{1}{3}\right)$$

$$\therefore \quad y_2 = \frac{35}{6}-5x+\frac{5x^2}{2}-\frac{x^3}{3}$$

$$y_3 = 3+\int_1^x (2x-y_2)\, dx$$

$$= 3+\int_1^x \left(\frac{-35}{6}+7x-\frac{5x^2}{2}+\frac{x^3}{3}\right) dx$$

$$= 3+\left[\frac{-35x}{6}+\frac{7x^2}{2}-\frac{5x^3}{6}+\frac{x^4}{12}\right]_1^x$$

$$y_3 = 3 + \left(\frac{-35x}{6} + \frac{7x^2}{2} - \frac{5x^3}{6} + \frac{x^4}{12}\right) - \left(\frac{-35}{6} + \frac{7}{2} - \frac{5}{6} + \frac{1}{12}\right)$$

Thus $\quad y_3 = \frac{73}{12} - \frac{35x}{6} + \frac{7x^2}{2} - \frac{5x^3}{6} + \frac{x^4}{12}$

Also from this expression we obtain $y(1.1) = 2.9145$

1.22 Taylor's series method

Consider the initial value problem : $\frac{dy}{dx} = f(x, y)$ and $y(x_0) = y_0$.

The solution $y(x)$ is approximated to a power series in $(x - x_0)$ using Taylor's series. Then we can find the value of y for various values of x in the neighbourhood of x_0.

We have Taylor's series expansion of $y(x)$ about the point x_0 in the form :

$$y(x) = y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2!}\,y''(x_0) + \frac{(x-x_0)^3}{3!}\,y'''(x_0) + \cdots$$

Here $y'(x_0)$, $y''(x_0)$, ... denote the value of the derivatives $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, ... at x_0 which can be found by making use of the data.

WORKED PROBLEMS

11. *Use Taylor's series method to find y at* $x = 0.1, 0.2, 0.3$ *considering terms upto the third degree given that* $\frac{dy}{dx} = x^2 + y^2$ *and* $y(0) = 1$

>> Taylor's series expansion of $y(x)$ is given by

$$y(x) = y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2!}\,y''(x_0) + \frac{(x-x_0)^3}{3!}\,y'''(x_0) + \cdots$$

By data $x_0 = 0$, $y_0 = 1$ and $y' = x^2 + y^2$

$$\therefore \quad y(x) = y(0) + x\,y'(0) + \frac{x^2}{2!}\,y''(0) + \frac{x^3}{3!}\,y'''(0) \qquad \ldots(1)$$

We need to compute $y'(0)$, $y''(0)$, $y'''(0)$.

Consider $y' = x^2 + y^2$; $y'(0) = 0^2 + [y(0)]^2 = 0 + 1 = 1$

Differentiating y' w.r.t. x we have,

$$y'' = 2x + 2yy' \;;$$

$$y''(0) = (2)(0) + 2 \cdot y(0) \cdot y'(0) = (2)(1)(1) = 2$$

Differentiating y'' w.r.t. x we have,

$$y''' = 2 + 2[yy'' + (y')^2]$$

$$\therefore \quad y'''(0) = 2 + 2[(1)(2) + 1^2] = 8$$

Substituting these values in (1) we have,

$$y(x) = 1 + x \cdot 1 + \frac{x^2}{2} \cdot 2 + \frac{x^3}{6} \cdot 8$$

That is, $y(x) = 1 + x + x^2 + \dfrac{4x^3}{3}$

This is called as Taylor's series approximation upto the third degree and we need to put $x = 0.1,\ 0.2,\ 0.3$ in the same. Thus we have,

$$\mathbf{y(0.1)} = 1 + 0.1 + (0.1)^2 + \frac{4(0.1)^3}{3} = \mathbf{1.1113}$$

$$\mathbf{y(0.2)} = 1 + 0.2 + (0.2)^2 + \frac{4(0.2)^3}{3} = \mathbf{1.2507}$$

$$\mathbf{y(0.3)} = 1 + 0.3 + (0.3)^2 + \frac{4(0.3)^3}{3} = \mathbf{1.426}$$

---

12. *Find y at $x = 1.02$ correct to five decimal places given $dy = (xy - 1)\,dx$ and $y = 2$ at $x = 1$ applying Taylor's series method .*

>> Taylor's series expansion is given by

$$y(x) = y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2!}\,y''(x_0) + \frac{(x-x_0)^3}{3!}\,y'''(x_0) + \cdots$$

By data $x_0 = 1,\ y_0 = 2$ and $y' = \dfrac{dy}{dx} = xy - 1$

Since the number of derivatives for approximation is not specifically mentioned, we shall have the approximation upto third degree. Hence we have

$$y(x) = y(1) + (x-1)\,y'(1) + \frac{(x-1)^2}{2!}\,y''(1) + \frac{(x-1)^3}{3!}\,y'''(1) \qquad \ldots(1)$$

Consider $y' = xy - 1 \quad ; \quad y'(1) = (1)(2) - 1 = 1$

$y'' = xy' + y \quad ; \quad y''(1) = (1)(1) + 2 = 3$

$y''' = xy'' + y' + y' \quad ; \quad y'''(1) = (1)(3) + 1 + 1 = 5$

To find $y(1.02)$, we shall substitute these values along with $x = 1.02$ in (1).

$$y(1.02) = 2 + (1.02 - 1)1 + \frac{(1.02-1)^2}{2} \cdot 3 + \frac{(1.02-1)^3}{6} \cdot 5$$

$$= 2 + (0.02) + \frac{(0.02)^2(3)}{2} + \frac{(0.02)^3(5)}{6}$$

Thus $\mathbf{y(1.02) = 2.02061}$

---

13. *From Taylor's series method, find $y(0.1)$ considering upto fourth degree term if $y(x)$ satisfies the equation $\frac{dy}{dx} = x - y^2$, $y(0) = 1$*

>> Taylors series expansion is given by

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots$$

By data $x_0 = 0$, $y_0 = 1$, $y' = x - y^2$

$$\therefore \quad y(x) = y(0) + xy'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) + \frac{x^4}{4!} y^{(4)}(0) \quad \ldots (1)$$

Consider $y' = x - y^2$; $y'(0) = 0 - 1^2 = -1$

$y'' = 1 - 2yy'$; $y''(0) = 1 - (2)(1)(-1) = 3$

$y''' = 0 - 2[yy'' + (y')^2]$; $y'''(0) = -2[(1)(3) + (-1)^2] = -8$

$y^{(4)} = -2[yy''' + y''y' + 2y'y''] = -2[yy''' + 3y'y'']$ *(fourth derivative)*

$\therefore \quad y^{(4)}(0) = -2[(1)(-8) + (3)(-1)(3)] = 34$

To find $y(0.1)$, we shall substitute these values along with $x = 0.1$ in (1).

$$y(0.1) = 1 + (0.1)(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24}(34)$$

Thus $\mathbf{y(0.1) = 0.9138}$

---

14. *Use Taylor's series method to obtain a power series in $(x-4)$ for the equation $5x\frac{dy}{dx} + y^2 - 2 = 0$; $x_0 = 4$, $y_0 = 1$ and use it to find y at $x = 4.1, 4.2, 4.3$ correct to four decimal places.*

>> Taylor's series expansion is given by

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots$$

Since $x_0 = 4$, $y_0 = 1$ by data, the series becomes

$$y(x) = y(4) + (x-4)y'(4) + \frac{(x-4)^2}{2} y''(4) + \cdots \qquad \ldots (1)$$

Consider $5xy' + y^2 - 2 = 0$

Substituting the initial values we obtain, [**Note** : $y' = y'(x)$]

$$(5)(4)y'(4) + 1^2 - 2 = 0 \quad \text{or} \quad 20y'(4) = 1$$

$$\therefore \quad y'(4) = \frac{1}{20} = 0.05$$

Differentiating the given equation w.r.t. x ,

$$5[xy'' + y'] + 2yy' = 0 \quad [\textbf{Note}: y'' = y''(x)]$$

Substituting the initial values and the value of $y'(4)$ we have,

$$5[4y''(4) + 0.05] + (2)(1)(0.05) = 0$$

ie., $\quad 20y''(4) + 0.25 + 0.1 = 0$

$$\therefore \quad y''(4) = -\frac{0.35}{20} = -0.0175$$

Since the value of the second derivative itself is small enough we shall approximate Taylor's series as in (1) upto second degree terms only.

Substituting these values in (1) we have,

$$y(x) = 1 + (x-4)(0.05) + \frac{(x-4)^2}{2}(-0.0175)$$

We now find $y(4.1)$, $y(4.2)$ and $y(4.3)$ from this expression.

$$y(4.1) = 1 + (4.1-4)(0.05) + \frac{(4.1-4)^2}{2}(-0.0175)$$

Thus we have,

$$\mathbf{y(4.1)} = 1 + (0.1)(0.05) + \frac{(0.1)^2}{2}(-0.0175) = \mathbf{1.0049}$$

$$\mathbf{y(4.2)} = 1 + (0.2)(0.05) + \frac{(0.2)^2}{2}(-0.0175) = \mathbf{1.0097}$$

$$\mathbf{y(4.3)} = 1 + (0.3)(0.05) + \frac{(0.3)^2}{2}(-0.0175) = \mathbf{1.0142}$$

- -

15. *Use Taylor's series method to find* $y(4.1)$ *given that* $\dfrac{dy}{dx} = \dfrac{1}{x^2 + y}$ *and* $y(4) = 4$.

>> Taylor's series expansion is given by

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots$$

By data $y' = \dfrac{1}{x^2 + y}$; $x_0 = 4,\ y_0 = 4$

$$\therefore\ y(x) = y(4) + (x-4)y'(4) + \frac{(x-4)^2}{2!}y''(4) + \frac{(x-4)^3}{3!}y'''(4) \quad \ldots(1)$$

by approximating upto the third degree terms.

Consider $\quad y' = \dfrac{1}{x^2 + y}$ [**Note** : $y' = y'(x),\ y'' = y''(x)$ etc.]

or $\quad y'(x^2 + y) = 1 \quad \ldots(2)$

Substituting the initial values we have,

$$y'(4)[4^2 + 4] = 1 \quad \text{or} \quad y'(4) = \frac{1}{20} = 0.05$$

Differentiating (2) w.r.t. x,

$$y'(2x + y') + (x^2 + y)y'' = 0 \quad \ldots(3)$$

Substituting the initial values and the value of $y'(4)$ we have,

$$0.05\,[(2)(4) + 0.05] + [4^2 + 4]\,y''(4) = 0$$

ie., $\quad 0.05\,[8 + 0.05] + 20\,y''(4) = 0$

ie., $\quad 0.4025 + 20\,y''(4) = 0$

$\therefore \quad y''(4) = -0.020125$

We observe that the value of the derivatives are small enough and the third degree term can also be neglected. Substituting these values in (1) for computing $y(4.1)$ we have,

$$y(4.1) = 4 + (4.1 - 4)(0.05) + \frac{(4.1-4)^2}{2}(-0.020125)$$

Thus $\quad \mathbf{y(4.1) = 4.0049}$

---

16. *Use Taylor's series method to solve* $y' = x^2 + y$ *in the range* $0 \le x \le 0.2$ *by taking step size* $h = 0.1$ *given that* $y = 10$ at $x = 0$ *initially considering terms upto the fourth degree.*

>> In this problem, since the step size is specified as 0.1, the problem has to be done in two stages. We have to first find $y(0.1)$ and use this as the initial condition to compute $y(0.2)$.

Taylor's series expansion is given by

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0)$$
$$+ \frac{(x-x_0)^3}{3!}y'''(x_0) + \frac{(x-x_0)^4}{4!}y''''(x_0) + \cdots \qquad \ldots (1)$$

I Stage: By data $y' = x^2 + y$, $x_0 = 0$, $y_0 = 10$

$y'(0) = 0^2 + 10 = 10$ or $y'(0) = 10$

Differentiating y' w.r.t. x successively we have,

$y'' = 2x + y'$; $y''(0) = (2)(0) + y'(0) = 0 + 10 = 10$

$y''' = 2 + y''$; $y'''(0) = 2 + y''(0) = 2 + 10 = 12$

$y'''' = y'''$; $y''''(0) = 12$

With $x = 0.1$ and $x_0 = 0$, (1) becomes

$$y(0.1) = y(0) + (0.1)y'(0) + \frac{(0.1)^2}{2}y''(0)$$
$$+ \frac{(0.1)^3}{6}y'''(0) + \frac{(0.1)^4}{24}y''''(0)$$
$$= 10 + (0.1)10 + \frac{0.01}{2}(10) + \frac{0.001}{6}(12) + \frac{0.0001}{24}(12)$$

Thus $\mathbf{y(0.1) = 11.05205 \approx 11.052}$

II Stage : Now taking $x_0 = 0.1$, $y_0 = 11.052$ we have

$y' = x^2 + y$; $y'(0.1) = (0.1)^2 + 11.052 = 11.062$

$y'' = 2x + y'$; $y''(0.1) = 2(0.1) + 11.062 = 11.262$

$y''' = 2 + y''$; $y'''(0.1) = 2 + 11.262 = 13.262$

$y'''' = y'''$; $y''''(0.1) = 13.262$

With $x = 0.2$ and $x_0 = 0.1$, (1) becomes

$$y(0.2) = y(0.1) + (0.1)y'(0.1) + \frac{0.01}{2} y''(0.1) + \frac{0.001}{6} y'''(0.1) + \frac{0.0001}{24} y''''(0.1)$$

$$= 11.052 + (0.1)(11.062) + \frac{0.01}{2}(11.262) + \frac{0.001}{6}(13.262) + \frac{0.0001}{24}(13.262)$$

Thus $\mathbf{y(0.2) = 12.216776 \approx 12.2168}$

---

17. *Employ Taylor's series method to find y at x = 0.1 and 0.2 correct to four places of decimal in step size of 0.1 given the linear differential equation* $\frac{dy}{dx} - 2y = 3e^x$ *whose solution passes through the origin. Also find y (0.1) and (0.2) by analytical method.*

>> By data, $y' = 2y + 3e^x$ and $y(0) = 0$. That is $x_0 = 0,\ y_0 = 0$

Taylor's series expansion is given by

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \cdots \qquad \ldots(1)$$

Step - 1 : We shall compute $y(0.1)$

Consider $\quad y' = 2y + 3e^x \quad ; \quad y'(0) = 2(0) + 3e^0 = 3$

$\therefore \quad y'' = 2y' + 3e^x \quad ; \quad y''(0) = 2(3) + 3 = 9$

$\quad y''' = 2y'' + 3e^x \quad ; \quad y'''(0) = 2(9) + 3 = 21$

From (1) we have,

$$y(0.1) = y(0) + (0.1)y'(0) + \frac{(0.1)^2}{2} y''(0) + \frac{(0.1)^3}{6} y'''(0)$$

by considering terms upto third degree. Further we have

$$y(0.1) = 0 + (0.1)3 + \frac{0.01}{2}(9) + \frac{0.001}{6}(21)$$

Thus $\mathbf{y(0.1) = 0.3485}$

Step - 2 : We shall compute $y(0.2)$

Consider $y' = 2y + 3e^x$ and let $x_0 = 0.1,\ y_0 = 0.3485$

Now $\quad y'(0.1) = 2y(0.1) + 3e^{0.1} \quad ; \quad y'(0.1) = 4.0125$

$\quad y'' = 2y' + 3e^x$

$\quad y''(0.1) = 2(4.0125) + 3e^{0.1} \quad ; \quad y''(0.1) = 11.3405$

$$y''' = 2y'' + 3e^x \; ; \; y'''(0.1) = 2(11.3405) + 3e^{0.1} \; ; \; y'''(0.1) = 25.9965$$

We have from (1)

$$y(0.2) = y(0.1) + (0.1)\, y'(0.1) + \frac{0.01}{2}\, y''(0.1) + \frac{0.001}{6}\, y'''(0.1)$$

$$\therefore \quad y(0.2) = 0.3485 + (0.1)\,(4.0125) + \frac{0.01}{2}\,(11.3405) + \frac{0.001}{6}\,(25.9965)$$

Thus $\mathbf{y(0.2) = 0.8108}$

Solution by analytical method

$\frac{dy}{dx} - 2y = 3e^x$ is of the form $\frac{dy}{dx} + Py = Q$ where $P = -2,\ Q = 3e^x$

Solution : $y\, e^{\int P\,dx} = \int Q\, e^{\int P\,dx}\, dx + c$

ie., $y\, e^{-2x} = \int 3e^x\, e^{-2x}\, dx + c$

$y\, e^{-2x} = -3e^{-x} + c$ or $y = -3e^x + c\,e^{2x}$ is the general solution.

Applying the condition $y(0) = 0$, the general solution becomes

$0 = -3 + c$ or $c = 3$

$\therefore \quad y = 3(e^{2x} - e^x)$ is the befitting solution.

Thus $\mathbf{y(0.1) = 0.3487}$ and $\mathbf{y(0.2) = 0.8113}$ by the analytical method.

---

18. *Using Taylor's series method, obtain the values of y at $x = 0.1\,(0.1)\,0.3$ to four significant figures if y satisfies the equation $y'' = -xy$ given that $y' = 0.5$ and $y = 1$ when $x = 0$ taking the first five terms of the Taylor's series expansion.*

\>> Taylor's series expansion is given by

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2!}\, y''(x_0) + \cdots \qquad \ldots (1)$$

By data, $y'' = -xy$; $y(0) = 1,\ y'(0) = 0.5$

Consider $y'' = -xy$; $y''(0) = 0$

$y''' = -xy' - y$; $y'''(0) = -1$

$y^{(4)} = -xy'' - 2y'$; $y^{(4)}(0) = -1$

From (1), the first five terms of the Taylor's series expansion is given by

$$y(x) = y(0) + x\,y'(0) + \frac{x^2}{2!}\, y''(0) + \frac{x^3}{3!}\, y'''(0) + \frac{x^4}{24}\, y^{(4)}(0)$$

Now, $y(0.1) = 1 + (0.1)(0.5) - \frac{0.001}{6} - \frac{0.0001}{24} = 1.0498$

$y(0.2) = 1 + (0.2)(0.5) - \frac{0.008}{6} - \frac{0.0016}{24} = 1.0986$

$y(0.3) = 1 + (0.3)(0.5) - \frac{0.027}{6} - \frac{0.0081}{24} = 1.1452$

Thus $\mathbf{y(0.1) = 1.0498,\ y(0.2) = 1.0986,\ y(0.3) = 1.1452}$

EXERCISES

Use Picard's method to solve the following initial value problems

1. $\frac{dy}{dx} = 1 + xy;\ y(0) = 2$. Compute $y(0.1)$, $y(0.2)$ and $y(0.3)$ from the third approximation.
2. $\frac{dy}{dx} = 1 + y^2;\ x_0 = 0,\ y_0 = 0$. Perform 3 iterations. Obtain the analytical solution as a power series upto the term containing x^5.
3. $\frac{dy}{dx} = x + y^2 + 1,\ y(0) = 0$. Obtain the series approximation. upto the fifth degree terms.
4. $\frac{dy}{dx} = x + y^2;\ y(0) = 1$. Compute $y(0.2)$ from the second approximation.
5. $\frac{dy}{dx} = y - x^2,\ y(0) = 1$ in the range $0 \le x \le 0.2$ by considering the third approximation to the solution.

Use Taylor's series method to solve the following initial value problems.

6. $\frac{dy}{dx} = x - y,\ y(0) = 1$. Compute $y(0.1)$ considering terms upto fourth degree.
7. $\frac{dy}{dx} = x^2 y - 1,\ y(0) = 1$. Compute $y(0.03)$ using the expansion of y upto second degree terms.
8. $\frac{dy}{dx} = x y^{1/3},\ y(1) = 1$. Compute $y(1.1)$ & $y(1.2)$ by taking step size 0.1
9. $\frac{dy}{dx} = x + y;\ x = 1, y = 0$. Find the third order approximation of the solution and use it to compute $y(1.1), y(1.2)$ and $y(1.3)$.
10. $\frac{dy}{dx} = x^2 y - 1, y(0) = 1$ in the range $0 \le x \le 0.2$ taking step size 0.1

ANSWERS

1. $2+x+x^2+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{15}+\frac{x^6}{24}$; 2.1104, 2.2431, 2.4012

2. $x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{x^7}{63}$. Analytical solution : $y = \tan x = x+\frac{x^3}{3}+\frac{2x^5}{15}$, by expanding $\tan x$ as a Maclaurin's series.

3. $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{20}$

4. $1+x+\frac{3x^2}{2}+\frac{2x^3}{3}+\frac{x^4}{4}+\frac{x^5}{20}$; 1.2657

5. $1+x+\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\frac{x^5}{60}$; $y(0.1) = 1.1018$ and $y(0.2) = 1.2079$

6. 0.8373

7. 0.97

8. 1.107, 1.228

9. $y = (x-1) + (x-1)^2 + \frac{1}{2}(x-1)^3$; 0.1105, 0.244, 0.4035

10. 0.9003, 0.8023

1.23 Modified Euler's Method

Consider the initial value problem $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$.

We need to find y at $x_1 = x_0 + h$.

We first obtain $y(x_1) = y_1$ by applying ***Euler's formula*** and this value is regarded as the initial approximation for y_1 usually denoted by $y_1^{(0)}$ also called as the predicted value of y_1.

Euler's formula is given by

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

Since the accuracy is poor in this formula this value y_1 is successively improved (*corrected*) to the desired degree of accuracy by the following ***Modified Euler's formula,*** where the successive approximations are denoted by $y_1^{(1)}, y_1^{(2)}, y_1^{(3)}, \ldots$ etc.

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

Each of the succeeding approximation is better than the preceding ones. They are called corrected values. Euler's formula and modified Euler's formula jointly are also called as ***Euler's predictor and corrector formulae.***

WORKED PROBLEMS

19. *Given* $\frac{dy}{dx} = 1 + \frac{y}{x}$, $y = 2$ *at* $x = 1$, *find the approximate value of* y *at* $x = 1.4$ *by taking step size* $h = 0.2$ *applying modified Euler's method. Also find the value of* y *at* $x = 1.2$ *and* 1.4 *from the analytical solution of the equation.*

>> The problem has to be worked in two stages.

I Stage : $x_0 = 1,\ y_0 = 2,\ f(x, y) = 1 + (y/x),\ h = 0.2$

$$x_1 = x_0 + h = 1.2,\ y(x_1) = y_1 = y(1.2) = ?$$

Now, $f(x_0, y_0) = 1 + (2/1) = 3$

We have Euler's formula : $y_1^{(0)} = y_0 + hf(x_0, y_0)$ ...(1)

$\therefore \quad y_1^{(0)} = 2 + (0.2)(3) = 2.6$

Further we have modified Euler's formula :

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \qquad \text{...(2)}$$

$$= 2 + (0.1)\,[3 + (1 + y_1^{(0)}/x_1)]$$

$$= 2 + (0.1)\,[4 + y_1^{(0)}/1.2]$$

$$= 2 + (0.1)\,[4 + 2.6/1.2]$$

$\therefore \quad y_1^{(1)} = 2.6167$

Next approximation $y_1^{(2)}$ is got just by replacing the value of $y_1^{(1)}$ in place of $y_1^{(0)}$

Now, $y_1^{(2)} = 2 + (0.1)\,[4 + 2.6167/1.2] = 2.6181$

Again $y_1^{(3)} = 2 + (0.1)\,[4 + 2.6181/1.2] = 2.6182$. Also $y_1^{(4)} = 2.6182$

Thus $y\,(1.2) = \mathbf{2.6182}$

II Stage : We repeat the process by taking $y(1.2) = 2.6182$ as the initial condition.

$$x_0 = 1.2,\ y_0 = 2.6182\ ;\ f(x_0, y_0) = 1 + (y_0/x_0) = 3.1818$$
$$x_1 = x_0 + h = 1.4,\ y(x_1) = y_1 = y(1.4) = ?$$

We have from (1),

$$y_1^{(0)} = 2.6182 + (0.2)(3.1818) = 3.2546$$

Now from (2),

$$y_1^{(1)} = 2.6182 + (0.1)\left[3.1818 + (1 + y_1^{(0)}/x_1)\right]$$

ie., $$y_1^{(1)} = 2.6182 + (0.1)\left[4.1818 + 3.2546/1.4\right] = 3.2689$$

$$y_1^{(2)} = 2.6182 + (0.1)\left[4.1818 + 3.2689/1.4\right] = 3.2699$$

$$y_1^{(3)} = 2.6182 + (0.1)\left[4.1818 + 3.2699/1.4\right] = 3.2699$$

Thus $\mathbf{y(1.4) = 3.2699 \approx 3.27}$

Now, let us find the analytical solution of the equation :

$$\frac{dy}{dx} = 1 + \frac{y}{x} \quad \text{or} \quad \frac{dy}{dx} - \frac{y}{x} = 1$$

This is a linear d.e. of the form $\dfrac{dy}{dx} + Py = Q$ whose solution is given by

$$y\,e^{\int P dx} = \int Q\,e^{\int P dx}\,dx + c$$

Here $P = -1/x$ and $Q = 1$

$$\therefore \quad e^{\int P dx} = e^{-\int 1/x\,dx} = e^{-\log x} = 1/e^{\log x} = 1/x$$

The solution becomes, $y \cdot 1/x = \int 1 \cdot 1/x\,dx + c$

ie., $y/x = \log x + c$.

Applying the initial condition that $y = 2$ at $x = 1$ we have,

$$2/1 = \log 1 + c \ \therefore\ c = 2 \ \text{ since } \log 1 = 0$$

The solution now becomes,

$$y/x = \log x + 2 \quad \text{or} \quad y = x(\log x + 2)$$

This is the analytical solution of the given initial value problem.

Now by putting $x = 1.2$ and 1.4 we obtain,

$$y(1.2) = 1.2(\log_e 1.2 + 2) = 2.6188 \quad \& \quad y(1.4) = 1.4(\log_e 1.4 + 2) = 3.2711$$

Solutions are tabulated for comparison.

Solution $y(x)$	By modified Euler's method	By analytical method
y (1.2)	**2.6182**	2.6188
y (1.4)	**3.2699**	3.2711

20. *Using modified Euler's method find y at $x = 0.2$ given $\frac{dy}{dx} = 3x + \frac{1}{2}y$ with $y(0) = 1$ taking $h = 0.1$. Perform three iterations at each step.*

>> We need to find $y(0.2)$ by taking $h = 0.1$.

This implies that the problem has to be done in two stages.

I Stage : By data $x_0 = 0,\ y_0 = 1, h = 0.1,\ f(x, y) = 3x + (y/2)$

$$f(x_0, y_0) = 0.5,\quad x_1 = x_0 + h = 0.1$$

$$y(x_1) = y_1 = y(0.1) = ?$$

From Euler's formula : $y_1^{(0)} = y_0 + hf(x_0, y_0)$ we obtain

$$y_1^{(0)} = 1 + (0.1)(0.5) = 1.05$$

We have modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] \quad \textit{(first iteration)}$$

$$= 1 + \frac{0.1}{2}\left[0.5 + 3x_1 + \frac{y_1^{(0)}}{2}\right]$$

$$= 1 + 0.05\left[0.5 + 3(0.1) + \frac{y_1^{(0)}}{2}\right]$$

$$= 1 + 0.05\left[0.8 + \frac{y_1^{(0)}}{2}\right]$$

ie., $$y_1^{(1)} = 1 + 0.05\left[0.8 + \frac{1.05}{2}\right] = 1.06625$$

The second iterative value is got simply by replacing $y_1^{(0)}$ by $y_1^{(1)}$

That is by replacing 1.06625 in place of 1.05

$$\therefore \quad y_1^{(2)} = 1 + 0.05\left[0.8 + \frac{1.06625}{2}\right] = 1.0667$$

$$y_1^{(3)} = 1 + 0.05\left[0.8 + \frac{1.0667}{2}\right] = 1.0667$$

Thus $\mathbf{y(0.1) = 1.0667}$

II Stage : Now, let $x_0 = 0.1, y_0 = 1.0667$

We have $f(x,y) = 3x + (y/2)$

$$\therefore \quad f(x_0, y_0) = 3(0.1) + \frac{1.0667}{2} = 0.83335$$

$$x_1 = x_0 + h = 0.2 \ ; \ y_1 = y(x_1) = y(0.2) = ?$$

From Euler's formula we obtain

$$y_1^{(0)} = 1.0667 + 0.1\ (0.83335) = 1.15$$

Next, from modified Euler's formula,

$$y_1^{(1)} = 1.0667 + \frac{0.1}{2}\left[0.83335 + 3x_1 + \frac{y_1^{(0)}}{2}\right]$$

$$= 1.0667 + 0.05\left[0.83335 + 3(0.2) + \frac{y_1^{(0)}}{2}\right]$$

$$= 1.0667 + 0.05\left[1.43335 + \frac{1.15}{2}\right] = 1.1671$$

$$y_1^{(2)} = 1.0667 + 0.05\left[1.43335 + \frac{1.1671}{2}\right] = 1.1675$$

$$y_1^{(3)} = 1.0667 + 0.05\left[1.43335 + \frac{1.1675}{2}\right] = 1.1676$$

Thus $\mathbf{y(0.2) = 1.1676}$

---

21. *Using modified Euler's method find $y(0.2)$ correct to four decimal places solving the equation $\frac{dy}{dx} = x - y^2, y(0) = 1$ taking $h = 0.1$*

>> We shall first compute $y(0.1)$ and use this value to compute $y(0.2)$

I Stage : By data $x_0 = 0,\ y_0 = 1,\ h = 0.1,\ f(x,y) = x - y^2$

$$f(x_0, y_0) = 0 - 1^2 = -1,\ x_1 = x_0 + h = 0.1$$

$$y(x_1) = y_1 = y(0.1) = ?$$

From Euler's formula : $y_1^{(0)} = y_0 + hf(x_0, y_0)$ we obtain

$$y_1^{(0)} = 1 + (0.1)(-1) = 0.9$$

We have modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,\ y_0) + f(x_1,\ y_1^{(0)})\right]$$

$$= 1 + \frac{0.1}{2}\left[-1 + x_1 - \left\{y_1^{(0)}\right\}^2\right]$$

$$= 1 + 0.05\ [-1 + 0.1 - (0.9)^2]$$

$$= 1 + 0.05\ [-0.9 - (0.9)^2] \quad = 0.9145$$

$$y_1^{(2)} = 1 + 0.05\ [-0.9 - (0.9145)^2] = 0.9132$$

$$y_1^{(3)} = 0.9133$$

Thus $y(0.1) = \mathbf{0.9133}$

II Stage : Now, let $x_0 = 0.1$, $y_0 = 0.9133$. $f(x, y) = x - y^2$

$$f(x_0, y_0) = 0.1 - (0.9133)^2 = -0.7341$$

$$x_1 = x_0 + h = 0.2,\ y(x_1) = y(0.2) = ?$$

Substituting in the Euler's formula,

$$y_1^{(0)} = 0.9133 + (0.1)(-0.7341) = 0.8399$$

Now from the modified Euler's formula,

$$y_1^{(1)} = 0.9133 + \frac{0.1}{2}\left[-0.7341 + x_1 - \left\{y_1^{(0)}\right\}^2\right]$$

$$= 0.9133 + 0.05\ [-0.7341 + 0.2 - (0.8399)^2]$$

$$= 0.9133 + 0.05\ [-0.5341 - (0.8399)^2] = 0.8513$$

$$y_1^{(2)} = 0.9133 + 0.05\ [-0.5341 - (0.8513)^2] = 0.8504$$

$$y_1^{(3)} = 0.9133 + 0.05\ [-0.5341 - (0.8504)^2] = 0.8504$$

Thus $y(0.2) = \mathbf{0.8504}$

---

22. *Using modified Euler's method find* $y(20.2)$ *and* $y(20.4)$ *given that* $\frac{dy}{dx} = \log_{10}\left(\frac{x}{y}\right)$ *with* $y(20) = 5$ *taking* $h = 0.2$

\>> We shall first compute $y(20.2)$ and use this value to compute $y(20.4)$

I Stage : By data $x_0 = 20,\ y_0 = 5$ and $h = 0.2$

$$f(x,y) = \log_{10}\left(\frac{x}{y}\right) ;\ f(x_0, y_0) = \log_{10}(4) = 0.6021$$

$$x_1 = x_0 + h = 20.2,\ y(x_1) = y_1 = y(20.2) = ?$$

From Euler's formula : $y_1^{(0)} = y_0 + hf(x_0, y_0)$ we obtain

$$y_1^{(0)} = 5 + (0.2)(0.6021) = 5.1204$$

Next by modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 5 + \frac{0.2}{2}\left[0.6021 + \log_{10}\left(\frac{x_1}{y_1^{(0)}}\right)\right]$$

$$\therefore \quad y_1^{(1)} = 5 + 0.1\left[0.6021 + \log_{10}\left(\frac{20.2}{5.1204}\right)\right] = 5.1198$$

$$y_1^{(2)} = 5 + 0.1\left[0.6021 + \log_{10}\left(\frac{20.2}{5.1198}\right)\right] = 5.1198$$

Thus $\mathbf{y(20.2) = 5.1198}$

II Stage : Now, let $x_0 = 20.2,\ y_0 = 5.1198$

$$f(x,y) = \log_{10}\left(\frac{x}{y}\right) ;\ f(x_0, y_0) = 0.5961$$

$$x_1 = x_0 + h = 20.4,\ y(x_1) = y_1 = y(20.4) = ?$$

Substituting in the Euler's formula,

$$y_1^{(0)} = 5.1198 + (0.2)(0.5961) = 5.239$$

Now by modified Euler's formula,

$$y_1^{(1)} = 5.1198 + \frac{0.2}{2}\left[0.5961 + \log_{10}\left(\frac{x_1}{y_1^{(0)}}\right)\right]$$

$$= 5.1198 + 0.1\left[0.5961 + \log_{10}\left(\frac{20.4}{5.239}\right)\right] = 5.2384$$

$$y_1^{(2)} = 5.1198 + 0.1\left[0.5961 + \log_{10}\left(\frac{20.4}{5.2384}\right)\right] = 5.2385$$

Thus **$y(20.4) = 5.2385$**

---

23. *Use modified Euler's method to solve* $\frac{dy}{dx} = x + |\sqrt{y}|$ *in the range* $0 \le x \le 0.4$ *by taking* $h = 0.2$ *given that* $y = 1$ at $x = 0$ *initially*.

>> We need to compute $y(0.2)$ and $y(0.4)$ with $h = 0.2$

I Stage : By data $x_0 = 0,\ y_0 = 1,\ f(x, y) = x + \sqrt{y},\ h = 0.2$

where the modulus sign indicates that we have to take only the positive value of $\sqrt{y}$.

$$f(x_0, y_0) = 0 + \sqrt{1} = 1. \qquad x_1 = x_0 + h = 0.2$$

$$y(x_1) = y_1 = y(0.2) = ?$$

From Euler's formula : $y_1^{(0)} = y_0 + hf(x_0, y_0)$ we obtain

$$y_1^{(0)} = 1 + 0.2(1) = 1.2$$

We have modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.2}{2}\left[1 + x_1 + \sqrt{y_1^{(0)}}\right]$$

$$= 1 + 0.1\ [1 + 0.2 + \sqrt{1.2}\,] = 1.2295$$

$$y_1^{(2)} = 1 + 0.1\ [1.2 + \sqrt{1.2295}\,] = 1.2309$$

$$y_1^{(3)} = 1 + 0.1\ [1.2 + \sqrt{1.2309}\,] = 1.2309$$

Thus **$y(0.2) = 1.2309$**

II Stage : Now let $x_0 = 0.2,\ y_0 = 1.2309$

$$f(x, y) = x + \sqrt{y}\ ;\ f(x_0, y_0) = 0.2 + \sqrt{1.2309} = 1.3095$$

$$x_1 = x_0 + h = 0.4, \quad y(x_1) = y_1 = y(0.4) = ?$$

Substituting in the Euler's formula,

$$y_1^{(0)} = 1.2309 + 0.2\ (1.3095) = 1.4928$$

Next from modified Euler's formula,

$$y_1^{(1)} = 1.2309 + \frac{0.2}{2}\left[1.3095 + x_1 + \sqrt{y_1^{(0)}}\right]$$

$$= 1.2309 + 0.1\ [1.3095 + 0.4 + \sqrt{1.4928}] = 1.524$$

$$y_1^{(2)} = 1.2309 + 0.1\ [1.7095 + \sqrt{1.524}] = 1.5253$$

$$y_1^{(3)} = 1.2309 + 0.1\ [1.7095 + \sqrt{1.5253}] = 1.5254$$

Also $y_1^{(4)} = 1.5254$

Thus $\mathbf{y(0.4) = 1.5254}$

---

24. *Use modified Euler's method to compute $y(0.1)$ given that $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$ by taking $h = 0.05$ considering the accuracy upto two approximations in each step.*

>> We need to compute $y(0.05)$ first and use this value to compute $y(0.1)$.

I Stage : By data $x_0 = 0$, $y_0 = 1$, $f(x, y) = x^2 + y$, $h = 0.05$

$$f(x_0, y_0) = 0^2 + 1 = 1, \quad x_1 = x_0 + h = 0.05$$

$$y(x_1) = y_1 = y(0.05) = ?$$

From Euler's formula : $y_1^{(0)} = y_0 + hf(x_0 y_0)$ we obtain

$$y_1^{(0)} = 1 + (0.05)(1) = 1.05$$

Next by modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.05}{2}\left[1 + x_1^2 + y_1^{(0)}\right]$$

$$= 1 + 0.025\ [1 + (0.05)^2 + 1.05]$$

$$= 1 + 0.025\ [1.0025 + 1.05] = 1.0513$$

$$y_1^{(2)} = 1 + 0.025\ [1.0025 + 1.0513] = 1.0513$$

Thus $\mathbf{y(0.05) = 1.0513}$

II Stage: Now, let $x_0 = 0.05$, $y_0 = 1.0513$

$$f(x,y) = x^2 + y\ ;\ f(x_0, y_0) = (0.05)^2 + 1.0513 = 1.0538$$

$$x_1 = x_0 + h = 0.1,\ y(x_1) = y_1 = y(0.1) = ?$$

Substituting in the Euler's formula,

$$y_1^{(0)} = 1.0513 + 0.05(1.0538) = 1.104$$

Next from the modified Euler's formula,

$$y_1^{(1)} = 1.0513 + \frac{0.05}{2}\left[1.0538 + x_1^2 + y_1^{(0)}\right]$$

$$= 1.0513 + 0.025\,[1.0538 + (0.1)^2 + 1.104]$$

$$= 1.0513 + 0.025\ [1.0638 + 1.104] = 1.1055$$

$$y_1^{(2)} = 1.0513 + 0.025\ [1.0638 + 1.1055] = 1.1055$$

Thus the required $\mathbf{y(0.1) = 1.1055}$

---

25. *Using Euler's predictor and corrector formulae solve* $\frac{dy}{dx} = x + y$ *at* $x = 0.2$ *given that* $y(0) = 1$

>> We need to compute $y(0.2)$ and since the step size is not specified we shall take it to be 0.2 itself.

By data we have, $x_0 = 0$, $y_0 = 1$, $f(x,y) = x + y$ and $h = 0.2$ (*assumed*)

$$f(x_0, y_0) = 0 + 1 = 1\quad x_1 = x_0 + h = 0.2$$

$$y(x_1) = y_1 = y(0.2) = ?$$

We have Euler's formula (*predictor formula*)

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1 + (0.2)1 = 1.2$$

Next consider modified Euler's formula (*corrector formula*)

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.2}{2}\left[1 + x_1 + y_1^{(0)}\right]$$

$$= 1 + 0.1\ [1 + 0.2 + (1.2)] = 1.24$$

$$y_1^{(2)} = 1 + 0.1\ [1.2 + (1.24)] = 1.244$$

$$y_1^{(3)} = 1 + 0.1\ [1.2 + 1.244] = 1.2444$$

$$y_1^{(4)} = 1 + 0.1\ [1.2 + 1.2444] = 1.24444$$

Thus the required $y(0.2) = \mathbf{1.2444}$

Remark : *If we had worked the problem in two stages (Taking $h = 0.1$) we would have got more accurate answer. It may be noted that lesser is the step size, greater is the accuracy.*

---

26. *Using Euler's predictor and corrector formula compute $y(1.1)$ correct to five decimal places given that $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ and $y = 1$ at $x = 1$. Also find the analytical solution.*

\>> By data, $\frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x}$ or $\frac{dy}{dx} = \frac{1 - xy}{x^2}$

We have $f(x, y) = \frac{1 - xy}{x^2}$; $x_0 = 1$, $y_0 = 1$. Let us take $h = 0.1$

$$f(x_0, y_0) = 0,\ x_1 = x_0 + h = 1.1$$

$$y(x_1) = y_1 = y(1.1) = ?$$

From Euler's formula : $y_1^{(0)} = y_0 + h f(x_0, y_0)$, we obtain $y_1^{(0)} = 1$

We have modified Euler's formula,

$$y_1^{(0)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$= 1 + \frac{0.1}{2}\left[0 + \frac{1 - x_1\, y_1^{(0)}}{x_1^2}\right]$$

$$= 1 + 0.05\left[\frac{1 - 1.1\ (1)}{(1.1)^2}\right] = 0.9959$$

$$y_1^{(2)} = 1 + 0.05 \left[\frac{1 - 1.1\,(0.9959)}{(1.1)^2} \right] = 0.99605$$

$$y_1^{(3)} = 1 + 0.05 \left[\frac{1 - 1.1\,(0.99605)}{(1.1)^2} \right] = 0.99605$$

thus $\mathbf{y(1.1) = 0.99605}$

Analytical Solution

$\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ is of the form $\frac{dy}{dx} + Py = Q$ where $P = 1/x$ and $Q = 1/x^2$

whose solution is given by

$$y\, e^{\int P\,dx} = \int Q\, e^{\int P\,dx}\, dx + c$$

$$e^{\int P\,dx} = e^{\int 1/x\,dx} = e^{\log x} = x$$

Solution : $y \cdot x = \int \frac{1}{x^2} \cdot x\, dx + c$

ie., $xy = \log x + c$. Using $y(1) = 1$, $1 = \log 1 + c$ $\therefore$ $c = 1$.

The befitting solution is $xy = \log x + 1$ or $y = \frac{\log x + 1}{x}$

Now $y(1.1) = \frac{\log(1.1) + 1}{1.1} = 0.99574$

Thus $\mathbf{y(1.1) = 0.99574}$ is the analytical solution.

1.24 Runge - Kutta method of fourth order

Consider the initial value problem $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$. We need to find $y(x_0 + h)$ where h is the step size.

We have to first compute k_1, k_2, k_3, k_4 by the following formulae.

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\, y_0 + \frac{k_1}{2} \right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\, y_0 + \frac{k_2}{2} \right)$$

$$k_4 = hf\,(x_0 + h,\, y_0 + k_3)$$

The required $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

WORKED PROBLEMS

27. *Given* $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$ *compute* $y(0.2)$ *by taking* $h = 0.2$ *using Runge-Kutta method of fourth order. Also find the analytical solution.*

>> By data $f(x,y) = 3x + \frac{y}{2}$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = hf(x_0, y_0) = (0.2)f(0,1) = (0.2)\left[3 \times 0 + \frac{1}{2}\right] = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)\,f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1}{2}\right)$$

$$k_2 = (0.2)f(0.1, 1.05) = (0.2)\left[3 \times 0.1 + \frac{1.05}{2}\right] = 0.165$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)f\left(0 + \frac{0.2}{2}, 1 + \frac{0.165}{2}\right)$$

$$k_3 = (0.2)\,f\,(0.1,\ 1.0825) = (0.2)\left[3 \times 0.1 + \frac{1.0825}{2}\right] = 0.16825$$

$$k_4 = hf\,(x_0 + h, y_0 + k_3) = (0.2)f(0.2,\ 1.16825)$$

$$k_4 = (0.2)\left[3 \times 0.2 + \frac{1.16825}{2}\right] = 0.236825$$

We have $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$\therefore \quad y(0.2) = 1 + \frac{1}{6}(0.1 + 2\times 0.165 + 2\times 0.16825 + 0.236825)$$

Thus $\mathbf{y(0.2) = 1.1672208}$

We shall find the analytical solution of the given equation by writing in the form $\frac{dy}{dx} + Py = Q$ whose solution is $y\,e^{\int P dx} = \int Q\,e^{\int P dx}\,dx + c$

We have $\frac{dy}{dx} - \frac{y}{2} = 3x$. Here $P = \frac{-1}{2}$, $Q = 3x$; $e^{\int P dx} = e^{-x/2}$

The solution becomes $y\,e^{-x/2} = 3\int x\,e^{-x/2}\,dx + c.$

Integrating RHS by parts we have,

$$y\,e^{-x/2} = 3\left[x e^{-x/2}(-2) - \int e^{-x/2}(-2).1\,dx\right] + c$$

$$y\,e^{-x/2} = 3\left[-2\,x\,e^{-x/2} - 4e^{-x/2}\right] + c$$

Multiplying with $e^{x/2}$ we obtain, $y = -6x - 12 + c\,e^{x/2}$

Applying the initial condition that $y = 1$ at $x = 0$ the solution becomes,

$$1 = 0 - 12 + c \qquad \therefore \quad c = 13$$

The analytical solution of the initial value problem is given by,

$$y = -6x - 12 + 13\,e^{x/2}$$

Now by putting $x = 0.2$ we have

$$y(0.2) = -6(0.2) - 12 + 13\,e^{0.1} = 1.1672219$$

Thus $y(0.2) = 1.1672219$, **by analytical method.**

---

28. *Use fourth order Runge-Kutta method to solve* $(x + y)\dfrac{dy}{dx} = 1$, $y(0.4) = 1$ *at* $x = 0.5$ *correct to four decimal places.*

>> We have $\dfrac{dy}{dx} = \dfrac{1}{x+y}$ and $y = 1$ at $x = 0.4$

$$f(x,y) = \frac{1}{x+y}, \; x_0 = 0.4, \; y_0 = 1 \quad y(0.5) = ?$$

Here $x_0 + h = 0.5 \;\therefore\; h = 0.5 - x_0 = 0.5 - 0.4 = 0.1$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = hf(x_0, y_0) = (0.1)f(0.4, 1) = (0.1)\left[\frac{1}{0.4 + 1}\right] = 0.0714$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)\,f\left(0.4 + \frac{0.1}{2}, 1 + \frac{0.0714}{2}\right)$$

$$k_2 = (0.1)\,f(0.45, 1.0357) = (0.1)\left[\frac{1}{0.45 + 1.0357}\right] = 0.0673$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)f(0.45, 1.03365)$$

$$k_3 = (0.1)\left[\frac{1}{0.45 + 1.03365}\right] = 0.0674$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)\,f(0.5, 1.0674)$$

$$k_4 = (0.1)\left[\frac{1}{0.5 + 1.0674}\right] = 0.0638$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\therefore \quad y(0.5) = 1 + \frac{1}{6}\left[0.0714 + 2(0.0673) + 2(0.0674) + 0.0638\right]$

Thus $\mathbf{y(0.5) = 1.0674}$

---

29. *Using Runge-Kutta method of fourth order, find $y(0.2)$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$ taking $h = 0.2$*

>> By data $f(x,y) = \frac{y-x}{y+x}$, $x_0 = 0$, $y_0 = 1$, $h = 0.2$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = hf(x_0, y_0) = (0.2)f(0, 1) = (0.2)\left[\frac{1-0}{1+0}\right] = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)f(0.1, 1.1)$$

$$k_2 = (0.2)\left[\frac{1.1 - 0.1}{1.1 + 0.1}\right] = 0.1667$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)f(0.1, 1.0835)$$

$$k_3 = (0.2)\left[\frac{1.0835 - 0.1}{1.0835 + 0.1}\right] = 0.1662$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.2)f(0.2, 1.1662)$$

$$k_4 = (0.2)\left[\frac{1.1662 - 0.2}{1.1662 + 0.2}\right] = 0.1414$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\therefore \quad y(0.2) = 1 + \frac{1}{6}\left[0.2 + 2(0.1667) + 2(0.1662) + 0.1414\right]$

Thus $\mathbf{y(0.2) = 1.1679}$

---

30. *Use fourth order Runge-Kutta method to find y at* $x = 0.1$ *given that* $\frac{dy}{dx} = 3e^x + 2y$, $y(0) = 0$ *and* $h = 0.1$

>> By data, $f(x, y) = 3e^x + 2y$, $x_0 = 0, y_0 = 0$, $h = 0.1$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = hf(x_0, y_0) = (0.1)f(0,0) = (0.1)[3e^0 + 2\times 0] = 0.3$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)f(0.05, 0.15)$$

$$k_2 = (0.1)[3e^{0.05} + 2(0.15)] = 0.3454$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)f(0.05, 0.1727)$$

$$k_3 = (0.1)[3e^{0.05} + 2(0.1727)] = 0.3499$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)f(0.1, 0.3499)$$

$$k_4 = (0.1)[3e^{0.1} + 2(0.3499)] = 0.4015$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$\therefore \quad y(0.1) = 0 + \frac{1}{6}\left[0.3 + 2(0.3454) + 2(0.3499) + 0.4015\right]$$

Thus $\mathbf{y(0.1) = 0.3487}$

---

31. *Use fourth order Runge-Kutta method to compute* $y(1.1)$ *given that* $\frac{dy}{dx} = xy^{1/3}$, $y(1) = 1$

>> By data $f(x,y) = xy^{1/3}$, $x_0 = 1, y_0 = 1$

We need to compute $y(1.1)$. This implies that $x_0 + h = 1.1 \quad \therefore \quad h = 0.1$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = hf(x_0, y_0)$$

$$k_1 = (0.1)f(1,1) = (0.1)[(1)(1)^{1/3}] = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)f(1.05, 1.05)$$

$$k_2 = (0.1)\,[(1.05)(1.05)^{1/3}] = 0.1067$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)f(1.05, 1.05335)$$

$$k_3 = (0.1)[(1.05)(1.05335)^{1/3}] = 0.1068$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)f(1.1, 1.1068)$$

$$k_4 = (0.1)\,[(1.1)\,(1.1068)^{1/3}] = 0.1138$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$\therefore \qquad y(1.1) = 1 + \frac{1}{6}\left[0.1 + 2(0.1067) + 2(0.1068) + 0.1138\right]$$

Thus $\mathbf{y(1.1) = 1.1068}$

---

32. *Using Runge Kutta method of fourth order solve* $\frac{dy}{dx} + y = 2x$ *at* $x = 1.1$ *given that* $y = 3$ *at* $x = 1$ *initially.*

>> We have $\frac{dy}{dx} = 2x - y$, $x_0 = 1$, $y_0 = 3$

$$f(x,y) = 2x - y,\; x_0 + h = 1.1 \quad \therefore \quad h = 0.1$$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = hf(x_0, y_0) = (0.1)f(1, 3) = (0.1)[2(1) - 3)] = -0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)f\left(1.05, 3 - \frac{0.1}{2}\right)$$

$$k_2 = (0.1)f(1.05, 2.95) = (0.1)[2(1.05) - 2.95] = -0.085$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)f\left(1.05, 3 - \frac{0.085}{2}\right)$$

$$k_3 = (0.1)f(1.05, 2.9575) = (0.1)\,[2(1.05) - 2.9575] = -0.08575$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)f(1.1, 3 - 0.08575)$$

$k_4 = (0.1) f(1.1,\ 2.91425) = (0.1)\ [2(1.1) - 2.91425]$

$k_4 = -0.071425$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\therefore \quad y(1.1) = 3 + \frac{1}{6}\left[-0.1 + 2(-0.085) + 2(-0.08575) - 0.071425\right]$

Thus $\quad \mathbf{y(1.1) = 2.9145125 \approx 2.9145}$

---

33. *Using Runge-kutta method of fourth order find* $y(0.2)$ *for the equation* $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$ *taking* $h = 0.1$

>> The problem has to be done in two stages.

I Stage : $f(x, y) = \frac{y-x}{y+x},\ x_0 = 0,\ y_0 = 1,\ h = 0.1$

We shall first compute k_1, k_2, k_3, k_4.

$$k_1 = hf(x_0, y_0) = (0.1) f(0,1) = (0.1)\left[\frac{1-0}{1+0}\right] = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = (0.1)\ f(0.05,\ 1.05)$$

$$k_2 = (0.1)\left[\frac{1.05 - 0.05}{1.05 + 0.05}\right] = 0.091$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = (0.1)\ f(0.05,\ 1.0455)$$

$$k_3 = (0.1)\left[\frac{1.0455 - 0.05}{1.0455 + 0.05}\right] = 0.0909$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3) = (0.1)\ f(0.1,\ 1.0909)$$

$$k_4 = (0.1)\left[\frac{1.0909 - 0.1}{1.0909 + 0.1}\right] = 0.0832$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\therefore \quad y(0.1) = 1 + \frac{1}{6}(0.1 + 0.182 + 0.1818 + 0.0832)$

Thus $\quad \mathbf{y(0.1) = 1.091167 \approx 1.0912}$

II Stage : $f(x, y) = \dfrac{y-x}{y+x}$; $x_0 = 0.1, y_0 = 1.0912, h = 0.1$

Again by using the same formulae for k_1, k_2, k_3, k_4 we have,

$$k_1 = (0.1)\, f(0.1,\ 1.0912) = (0.1)\left[\frac{1.0912-0.1}{1.0912+0.1}\right] = 0.0832$$

$$k_2 = (0.1)\, f(0.15,\ 1.1328) = 0.0766$$

$$k_3 = (0.1)\, f(0.15,\ 1.1295) = 0.07655$$

$$k_4 = (0.1)\, f(0.2,\ 1.16775) = 0.07075$$

Now $\quad y(x_0+h) = y_0 + \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$y(0.1+0.1) = 1.0912 + \frac{1}{6}(0.0832 + 0.1532 + 0.1531 + 0.07075)$$

Thus $\quad$ **$y(0.2) = 1.167908 \approx 1.1679$**

Remark: *Referring to Problem-29, the problem has been worked in one stage with* $h = 0.2$ *and we have obtained* $y(0.2) = 1.1679$

---

34. *Solve :* $(y^2 - x^2)\, dx = (y^2 + x^2)\, dy$ *for* $x = 0\,(0.2)\,0.4$ *given that* $y = 1$ *at* $x = 0$ *initially, by applying Runge - Kutta method of order 4.*

>> We have $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$; $x_0 = 0, y_0 = 1, h = 0.2$

I Stage : $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}$ We shall compute k_1, k_2, k_3, k_4.

$$k_1 = hf(x_0, y_0) = (0.2)\, f(0, 1) = (0.2)\,1 = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) = (0.2)\, f(0.1,\ 1.1) = 0.1967$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) = (0.2)\, f(0.1,\ 1.0984) = 0.1967$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3) = (0.2)\, f(0.2,\ 1.1967) = 0.1891$$

We have, $y(x_0 + h) = y_0 + \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Thus $y(0.2) = 1.19598 \approx 1.196$

II Stage : $f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$, $x_0 = 0.2$, $y_0 = 1.196$, $h = 0.2$

Again using the same formula for k_1, k_2, k_3, k_4 we have,

$$k_1 = (0.2) f(0.2, 1.196) = 0.1891$$

$$k_2 = (0.2) f(0.3, 1.29055) = 0.1795$$

$$k_3 = (0.2) f(0.3, 1.28575) = 0.1793$$

$$k_4 = (0.2) f(0.4, 1.3753) = 0.1688$$

Now $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Thus $\mathbf{y(0.4) = 1.37525 \approx 1.3753}$

EXERCISES

Apply modified Euler's method to solve the following initial value problems by considering the accuracy upto two approximations in every step.

1. $\frac{dy}{dx} = -xy^2$, $y(0) = 2$. Compute $y(0.2)$ by taking $h = 0.1$

2. $\frac{dy}{dx} + y = 1, y = 0$ at $x = 0$. Compute y at $x = 0.4$ by taking $h = 0.1$

3. $\frac{dy}{dx} = x^2 + y$ in the range $0 \le x \le 0.06$ by taking $h = 0.02$ given that $y = 1$ at $x = 0$ initially.

4. $\frac{dy}{dx} = x + y$, $y(0) = 1$. Compute $y(0.2)$ taking $h = 0.1$

5. $\frac{dy}{dx} + x^2 = y$, $y(0) = 1$. Compute y in the range $[0, 0.6]$ by taking $h = 0.2$.

6. $\frac{dy}{dx} = 2 + |\sqrt{xy}|$, $y = 1$ at $x = 1$. Compute y at $x = 2$ by taking $h = 0.2$.

7. $\frac{dy}{dx} = x + |\sqrt{y}|$, $y(0) = 1$. Compute $y(0.6)$ by taking $h = 0.2$

Use fourth order Runge-Kutta method to solve the following initial value problems.

8. $\frac{dy}{dx} = x + y,\ y(0) = 1$. Compute $y(0.2)$

9. $\frac{dy}{dx} = x + y^2,\ y(0) = 1$. Compute $y(0.2)$ by taking $h = 0.1$

10. $\frac{dy}{dx} = 3x + \frac{y}{2},\ y(0) = 1$. Compute $y(0.2)$ with $h = 0.1$

11. $10\frac{dy}{dx} = x^2 + y^2,\ y(0) = 1$ for the interval $0 \le x \le 0.2$ by taking $h = 0.1$.

12. $\frac{dy}{dx} = xy,\ y(1) = 2$. Compute $y(1.2)$ by taking $h = 0.2$.

13. $\frac{dy}{dx} = 1 + y^2,\ x_0 = 0, y_0 = 0$. Compute the values of y corresponding to the values of x in the range $0 \le x \le 0.6$ by taking $h = 0.2$

14. $\frac{dy}{dx} = x(1 + xy),\ y(0) = 1$ in the range $0 \le x \le 0.2$ by taking $h = 0.1$

15. $\frac{dy}{dx} = \log_{10}(x/y)$ in the range $20 \le x \le 20.4$ by taking $h = 0.2$ given that $y = 5$ when $x = 20$.

ANSWERS

1. 1.923
2. 0.3292
3. 1.0202, 1.0408, 1.0619
4. 1.2427
5. 1.218, 1.467, 1.737
6. 5.051
7. 1.8851
8. 1.2428
9. 1.2736
10. 1.1678
11. 1.0101, 1.0205
12. 2.4921
13. 0.2027, 0.4228, 0.6841
14. 1.0053, 1.0227
15. 5.12, 5.24

1.3 Predictor and Corrector methods

In these methods the value of y at a desired vale of x is estimated from a set of four values of y corresponding to four equally spaced values of x. The four values may be readily available or be generated using the given initial condition by any numerical method discussed earlier. Picards method or Taylor's series method would be appropriate to generate three more values of y given one value initially.

We discuss two predictor and corrector methods namely,

1. Milne's method 2. Adams - Bashforth method

Consider the differential equation $y' = \frac{dy}{dx} = f(x, y)$ with a set of four pre determined values of $y : y(x_0) = y_0, y(x_1) = y_1, y(x_2) = y_2$ and $y(x_3) = y_3$.

Here x_0, x_1, x_2, x_3 are equally spaced values of x with width h.

Also $x_4 = x_3 + h = x_0 + 4h$

Predictor and Corrector formulae to compute $y(x_4) = y_4$ are as follows.

1.31 Milne's predictor and corrector formulae

$$y_4^{(P)} = y_0 + \frac{4h}{3} (2y_1' - y_2' + 2y_3') \ldots \quad (\textit{Predictor formula})$$

$$y_4^{(C)} = y_2 + \frac{h}{3} (y_2' + 4y_3' + y_4') \ldots \quad (\textit{Corrector formula})$$

Note : *These two formulae can be written in the general form as follows :*

$$y_{n+1}^{(P)} = y_{n-3} + \frac{4h}{3} \left[2y'_{n-2} - y'_{n-1} + 2y_n' \right]$$

$$y_{n+1}^{(C)} = y_{n-1} + \frac{h}{3} \left[y'_{n-1} + 4y'_n + y'_{n+1} \right]$$

1.32 Adams - Bashforth predictor and corrector formulae

$$y_4^{(P)} = y_3 + \frac{h}{24} (55y_3' - 59y_2' + 37y_1' - 9y_0') \quad (\textit{Predictor formula})$$

$$y_4^{(C)} = y_3 + \frac{h}{24} (9y_4' + 19y_3' - 5y_2' + y_1') \quad (\textit{Corrector formula})$$

Note : *We can write down the general form of these two formulae also as in Milne's method.*

Working procedure for problems

- We first prepare the table showing the values of y corresponding to four equidistant values of x and the computation of $y' = f(x, y)$.
- We compute y_4 from the predictor formula.
- We use this value of y_4 to compute $y_4' = f(x_4, y_4)$
- We apply corrector formula to obtain the corrected value of y_4.
- This value is used for computing y_4' to apply the corrector formula again.
- The process is continued till we get consistency in two consecutive values of y_4.

Note : *We can also find $y_5, y_6 \ldots$ by deducing expressions from the general form of the predictor and corrector formulae.*

WORKED PROBLEMS

35. *Given that* $\dfrac{dy}{dx} = x - y^2$ *and the data* $y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$. *Compute* y *at* $x = 0.8$ *by applying*

(a) *Milne's method* (b) *Adams - Bashforth method*

>> We prepare the following table using the given data which is essentially required for applying the predictor and corrector formulae.

x	y	$y' = x - y^2$
$x_0 = 0$	$y_0 = 0$	$y_0' = 0 - 0^2 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$y_1' = 0.2 - (0.02)^2 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$y_2' = 0.4 - (0.0795)^2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$y_3' = 0.6 - (0.1762)^2 = 0.5689$
$x_4 = 0.8$	$y_4 = ?$	

(a) By Milne's method

We have the predictor formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

$$\therefore \quad y_4^{(P)} = 0 + \frac{4(0.2)}{3}[2(0.1996) - 0.3937 + 2(0.5689)] = 0.3049$$

Now $y_4' = x_4 - y_4^2 = 0.8 - (0.3049)^2 = 0.707$

Next we have the corrector formula,

$$y_4^{(C)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

$$\therefore \quad y_4^{(C)} = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.707] = 0.3046$$

Now, $y_4' = x_4 - y_4^2 = 0.8 - (0.3046)^2 = 0.7072$

Substituting this value of y_4' again in the corrector formula,

$$y_4^{(C)} = 0.0795 + \frac{0.2}{3}\left[0.3937 + 4(0.5689) + 0.7072\right] = 0.3046$$

Thus $\mathbf{y_4 = y(0.8) = 0.3046}$

(b) By Adams - Bashforth method

We have the predictor formula,

$$y_4^{(P)} = y_3 + \frac{h}{24}\,(55\,y_3' - 59\,y_2' + 37\,y_1' - 9\,y_0')$$

$$\therefore \quad y_4^{(P)} = 0.1762 + \frac{0.2}{24}\left[55\,(0.5689) - 59\,(0.3937) + 37\,(0.1996) - 9\,(0)\right]$$

$$y_4^{(P)} = 0.3049$$

Now $y_4' = x_4 - y_4^2 = 0.8 - (0.3049)^2 = 0.707$

Next, we have the corrector formula,

$$y_4^{(C)} = y_3 + \frac{h}{24}\,(9\,y_4' + 19\,y_3' - 5\,y_2' + y_1')$$

$$\therefore \quad y_4^{(C)} = 0.1762 + \frac{0.2}{24}\left[9\,(0.707) + 19\,(0.5689) - 5\,(0.3937) + 0.1996\right]$$

$$y_4^{(C)} = 0.3046$$

Now $y_4' = x_4 - y_4^2 = 0.8 - (0.3046)^2 = 0.7072$

Applying the corrector formulae again with only change in the value of y_4' we obtain,

$$y_4^{(C)} = 0.3046$$

Thus $\boldsymbol{y_4 = y(0.8) = 0.3046}$

---

36. *Apply Milne's method to compute $y(1.4)$ correct to four decimal places given $\frac{dy}{dx} = x^2 + \frac{y}{2}$ and following the data :*

$$y(1) = 2,\; y(1.1) = 2.2156,\; y(1.2) = 2.4649 \quad y(1.3) = 2.7514$$

>> First we shall prepare the following table.

x	y	$y' = x^2 + \frac{y}{2}$
$x_0 = 1$	$y_0 = 2$	$y_0' = 1^2 + \frac{2}{2} = 2$
$x_1 = 1.1$	$y_1 = 2.2156$	$y_1' = (1.1)^2 + \frac{2.2156}{2} = 2.3178$
$x_2 = 1.2$	$y_2 = 2.4649$	$y_2' = (1.2)^2 + \frac{2.4649}{2} = 2.67245$
$x_3 = 1.3$	$y_3 = 2.7514$	$y_3' = (1.3)^2 + \frac{2.7514}{2} = 3.0657$
$x_4 = 1.4$	$y_4 = ?$	

We have $y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$

$\therefore \quad y_4^{(P)} = 2 + \frac{4(0.1)}{3}[2(2.3178) - 2.67245 + 2(3.0657)] = 3.0793.$

Hence $y_4' = x_4^2 + \frac{y_4}{2} = (1.4)^2 + \frac{3.0793}{2} = 3.49965$

Now consider $y_4^{(C)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$

$\therefore \quad y_4^{(C)} = 2.4649 + \frac{0.1}{3}[2.67245 + 4(3.0657) + 3.49965] = 3.0794.$

Now $y_4' = (1.4)^2 + \frac{3.0794}{2} = 3.4997$

Substituting this value of y_4' again in the corrector formula we obtain $y_4^{(C)} = 3.0794$

Thus $\quad \mathbf{y_4 = y(1.4) = 3.0794}$

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37. *Use Taylor's series method (upto third derivative term) to find y at* $x = 0.1, 0.2, 0.3$ *given that* $\frac{dy}{dx} = x^2 + y^2$ *with* $y(0) = 1$.

Apply Milne's predictor-corrector formulae to find $y(0.4)$ *using the generated set of initial values.*

>> Referring to Problem - 11, we have obtained $y(0.1) = 1.1113$, $y(0.2) = 1.2507$ $y(0.3) = 1.426$. Using these values along with $y(0) = 1$ initially, we prepare the following table.

x	y	$y' = x^2 + y^2$
$x_0 = 0$	$y_0 = 1$	$y_0' = 0^2 + 1^2 = 1$
$x_1 = 0.1$	$y_1 = 1.1113$	$y_1' = (0.1)^2 + (1.1113)^2 = 1.245$
$x_2 = 0.2$	$y_2 = 1.2507$	$y_2' = (0.2)^2 + (1.2507)^2 = 1.6043$
$x_3 = 0.3$	$y_3 = 1.426$	$y_3' = (0.3)^2 + (1.426)^2 = 2.1235$
$x_4 = 0.4$	$y_4 = ?$	

Consider $y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$

$\therefore \quad y_4^{(P)} = 1 + \frac{4(0.1)}{3}\left[2(1.245) - 1.6043 + 2(2.1235)\right] = 1.6844$

Hence $y'_4 = x_4^2 + y_4^2 = (0.4)^2 + (1.6844)^2 = 2.9972$

Next we have, $y_4^{(C)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$

$$\therefore \quad y_4^{(C)} = 1.2507 + \frac{0.1}{3}\left[1.6043 + 4(2.1235) + 2.9972\right] = 1.6872$$

Now $y_4' = (0.4)^2 + (1.6872)^2 = 3.0067$

Substituting this value of y_4' in the corrector formula again,

$$y_4^{(C)} = 1.2507 + \frac{0.1}{3}[1.6042 + 4(2.1235) + 3.0067] = 1.6875$$

Now $y_4' = (0.4)^2 + (1.6875)^2 = 3.0077$

Substituting again in the corrector formula we obtain $y_4^{(C)} = 1.6876$

Now $y_4' = (0.4)^2 + (1.6876)^2 = 3.008$

Substituting again in the corrector formula we obtain $y_4^{(C)} = 1.6875733 \approx 1.6876$

Thus **$y(0.4) = 1.6876$**

---

38. *The following table gives the solution of* $5xy' + y^2 - 2 = 0$. *Find the value of y at* $x = 4.5$ *using Milne's predictor and corrector formulae. Use the corrector formula twice.*

x	4	4.1	4.2	4.3	4.4
y	1	1.0049	1.0097	1.0143	1.0187

>> By data $5xy' + y^2 - 2 = 0$ or $y' = \frac{2 - y^2}{5x}$

We prepare the following table.

x	y	$y' = \frac{2-y^2}{5x}$
$x_0 = 4$	$y_0 = 1$	$y_0' = \frac{2-1^2}{5\times4} = 0.05$
$x_1 = 4.1$	$y_1 = 1.0049$	$y_1' = \frac{2-(1.0049)^2}{5\times4.1} = 0.0483$
$x_2 = 4.2$	$y_2 = 1.0097$	$y_2' = \frac{2-(1.0097)^2}{5\times4.2} = 0.0467$
$x_3 = 4.3$	$y_3 = 1.0143$	$y_3' = \frac{2-(1.0143)^2}{5\times4.3} = 0.0452$
$x_4 = 4.4$	$y_4 = 1.0187$	$y_4' = \frac{2-(1.0187)^2}{5\times4.4} = 0.0437$
$x_5 = 4.5$	$y_5 = ?$	

We have Milne's predictor and corrector formulae in the standard form

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')\ ;$$

$$y_4^{(C)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

Since we require y_5, the equivalent form of these formulae is given by

$$y_5^{(P)} = y_1 + \frac{4h}{3}(2y_2' - y_3' + 2y_4')\ ;$$

$$y_5^{(C)} = y_3 + \frac{h}{3}(y_3' + 4y_4' + y_5')$$

Hence $y_5^{(P)} = 1.0049 + \frac{4(0.1)}{3}[2(0.0467) - 0.0452 + 2(0.0437)] = 1.023$

Now $y_5' = \frac{2-y_5^2}{5x_5} = \frac{2-(1.023)^2}{5\times4.5} = 0.0424$

Hence, $y_5^{(C)} = 1.0143 + \frac{0.1}{3}[0.0452 + 4(0.0437) + 0.0424] = 1.023$

Thus $\mathbf{y(4.5) = 1.023}$

Remark : *Though we had a set of five pre determined values of y, we used only a set of four values to determine the fifth value in the sequence.*

39. *If* $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.040$ *and* $y(0.3) = 2.090$, *find* $y(0.4)$ *correct to four decimal places by using*

(a) *Milne's predictor-corrector method.*

(b) *Adams - Bashforth predictor - corrector method. (Apply the corrector formula twice)*

>> We prepare the following table.

x	y	$y' = 2e^x - y$
$x_0 = 0$	$y_0 = 2$	$y_0' = 2e^0 - 2 = 0$
$x_1 = 0.1$	$y_1 = 2.010$	$y_1' = 2e^{0.1} - 2.01 = 0.2003$
$x_2 = 0.2$	$y_2 = 2.040$	$y_2' = 2e^{0.2} - 2.04 = 0.4028$
$x_3 = 0.3$	$y_3 = 2.090$	$y_3' = 2e^{0.3} - 2.09 = 0.6097$
$x_4 = 0.4$	$y_4 = ?$	

(a) By Milne's predictor-corrector method

We have Milne's predictor formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

$$\therefore \quad y_4^{(P)} = 2 + \frac{4(0.1)}{3}[2(0.2003) - 0.4028 + 2(0.6097)] = 2.1623$$

Now $y_4' = 2e^{0.4} - 2.1623 = 0.8213$

We have Milne's corrector formula,

$$y_4^{(C)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

$$\therefore \quad y_4^{(C)} = 2.04 + \frac{0.1}{3}[0.4028 + 4(0.6097) + 0.8213] = 2.1621$$

Now $y_4' = 2e^{0.4} - 2.1621 = 0.8215$

Applying the corrector formula again we have,

$$y_4^{(C)} = 2.04 + \frac{0.1}{3}[0.4028 + 4(0.6097) + 0.8215] = 2.1621$$

Thus $y(0.4) = \mathbf{2.1621}$

(b) By Adams - Bashforth predictor - corrector method

We have, $y_4^{(P)} = y_3 + \frac{h}{24}(55\,y_3' - 59\,y_2' + 37\,y_1' - 9\,y_0')$

$$\therefore \quad y_4^{(P)} = 2.09 + \frac{0.1}{24}\left[55(0.6097) - 59(0.4028) + 37(0.2003) - 9(0)\right] = 2.1616$$

Now, $y_4' = 2e^{0.4} - 2.1616 = 0.822$

Next we have, $y_4^{(C)} = y_3 + \frac{h}{24}(9\,y_4' + 19\,y_3' - 5\,y_2' + y_1')$

$$\therefore \quad y_4^{(C)} = 2.09 + \frac{0.1}{24}\left[9(0.822) + 19(0.6097) - 5(0.4028) + 0.2003\right] = 2.1615$$

Now, $y_4' = 2e^{0.4} - 2.1615 = 0.82215$

Substituting again in the corrector formula, we obtain $y_4^{(C)} = 2.1615$

Thus $\mathbf{y(0.4) = 2.1615}$

---

40. *Apply Adams - Bashforth method to solve the equation $(y^2+1)\,dy - x^2\,dx = 0$ at $x = 1$ given $y(0) = 1$ by generating the initial values from Picard's second approximation at a step size of 0.25. Apply the corrector formula twice.*

>> By data, $\frac{dy}{dx} = y' = \frac{x^2}{y^2+1}$

Referring to Problem-5 for the values $y(0.25), y(0.5)$ and $y(0.75)$ we prepare the following table.

x	y	$y' = \frac{x^2}{y^2+1}$
$x_0 = 0$	$y_0 = 1$	$y_0' = \frac{0^2}{1^2+1} = 0$
$x_1 = 0.25$	$y_1 = 1.0026$	$y_1' = \frac{(0.25)^2}{(1.0026)^2+1} = 0.0312$
$x_2 = 0.5$	$y_2 = 1.0206$	$y_2' = \frac{(0.5)^2}{(1.0206)^2+1} = 0.1225$
$x_3 = 0.75$	$y_3 = 1.0679$	$y_3' = \frac{(0.75)^2}{(1.0679)^2+1} = 0.2628$
$x_4 = 1$	$y_4 = ?$	

We have the predictor formula

$$y_4^{(P)} = y_3 + \frac{h}{24}\,(55\,y_3' - 59\,y_2' + 37\,y_1' - 9\,y_0')$$

$$\therefore \quad y_4^{(P)} = 1.0679 + \frac{0.25}{24}\,[\,55\,(0.2628) - 59\,(0.1225) + 37\,(0.0312) - 9\,(0)\,] = 1.1552$$

Now $\quad y_4' = \dfrac{x_4^2}{y_4^2 + 1} = \dfrac{1^2}{(1.1552)^2 + 1} = 0.4284$

Next we have the corrector formula,

$$y_4^{(C)} = y_3 + \frac{h}{24}\,(9\,y_4' + 19\,y_3' - 5\,y_2' + y_1')$$

$$y_4^{(C)} = 1.0679 + \frac{0.25}{24}\,[\,9\,(0.4284) + 19\,(0.2628) - 5\,(0.1224) + 0.0312\,]\,y_4^{(C)} = 1.154$$

Now, $\quad y_4' = \dfrac{1^2}{(1.154)^2 + 1} = 0.4289$

Applying the corrector formula again we obtain, $y_4^{(C)} = 1.1541$

Thus the required $y(1) = \mathbf{1.1541}$

---

41. *Find the value of y at $x = 4.4$ by applying Adams - Bashforth method given that $5x\,\dfrac{dy}{dx} + y^2 - 2 = 0$ and $y = 1$ at $x = 4$ initially by generating the other required values from the Taylor's polynomial.*

\>> We need to generate the value of y at $x = 4.1,\ 4.2,\ 4.3$

[*Refer Problem-14 for the generation of the required values*]

We have obtained $y(4.1) = 1.0049,\ y(4.2) = 1.0097,\ y(4.3) = 1.0142$ by using the given initial condition $y(4) = 1$.

We prepare the following table.

x	y	$y' = \dfrac{2-y^2}{5x}$
$x_0 = 4$	$y_0 = 1$	$y_0' = \dfrac{2-1^2}{5 \times 4} = 0.05$
$x_1 = 4.1$	$y_1 = 1.0049$	$y_1' = \dfrac{2-(1.0049)^2}{5 \times 4.1} = 0.0483$
$x_2 = 4.2$	$y_2 = 1.0097$	$y_2' = \dfrac{2-(1.0097)^2}{5 \times 4.2} = 0.0467$
$x_3 = 4.3$	$y_3 = 1.0142$	$y_3' = \dfrac{2-(1.0142)^2}{5 \times 4.3} = 0.0452$
$x_4 = 4.4$	$y_4 = ?$	

We have the predictor formula,

$$y_4^{(P)} = y_3 + \frac{h}{24}\,(55\,y_3' - 59\,y_2' + 37\,y_1' - 9\,y_0')$$

$$y_4^{(P)} = 1.0142 + \frac{0.1}{24}\,[\,55\,(0.0452) - 59\,(0.0467) + 37\,(0.0483) - 9\,(0.05)\,]$$

$\therefore \quad y_4^{(P)} = 1.0187$

Now $\quad y_4' = \dfrac{2-y_4^2}{5x_4} = \dfrac{2-(1.0187)^2}{5 \times 4.4} = 0.0437$

Next we have, $\quad y_4^{(C)} = y_3 + \dfrac{h}{24}\,(9\,y_4' + 19\,y_3' - 5\,y_2' + y_1')$

$$y_4^{(C)} = 1.0142 + \frac{0.1}{24}\,[\,9\,(0.0437) + 19\,(0.0452) - 5\,(0.0467) + 0.0485\,]$$

$\therefore \quad y_4^{(C)} = 1.0186$

Now $\quad y_4' = \dfrac{2-(1.0186)^2}{5 \times 4.4} = 0.0437$ (*Same value as earlier*)

Thus $\quad$ **$y(4.4) = 1.0186$**

---

42. *Solve the differential equation* $y' + y + x y^2 = 0$ *with the initial values of* y : $y_0 = 1$, $y_1 = 0.9008$, $y_2 = 0.8066$, $y_3 = 0.722$ *corresponding to the values of* $x : x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3$ *by computing the value of* y *corresponding to* $x = 0.4$ *applying Adams - Bashforth predictor and corrector formula.*

>> We prepare the following table.

x	y	$y' = -(y + x y^2)$
$x_0 = 0$	$y_0 = 1$	$y_0' = -1$
$x_1 = 0.1$	$y_1 = 0.9008$	$y_1' = -0.9819$
$x_2 = 0.2$	$y_2 = 0.8066$	$y_2' = -0.9367$
$x_3 = 0.3$	$y_3 = 0.722$	$y_3' = -0.8784$
$x_4 = 0.4$	$y_4 = ?$	

We have the predictor formula,

$$y_4^{(P)} = y_3 + \frac{h}{24}\,(55\,y_3' - 59\,y_2' + 37\,y_1' - 9\,y_0')$$

On substitution we obtain, $y_4^{(P)} = 0.6371$

Now $y_4' = -[\,y_4 + x_4 y_4^2\,] = -0.7995$

Next we have, $y_4^{(C)} = y_3 + \dfrac{h}{24}\,(9\,y_4' + 19\,y_3' - 5\,y_2' + y_1')$

On substitution we obtain, $y_4^{(C)} = 0.6379$

Now $y_4' = -[\,0.6379 + (0.4)\,(0.6379)^2\,] = -0.8007$

Applying the corrector formula again we obtain $y_4^{(C)} = 0.6379$

Thus $y(0.4) = \mathbf{0.6379}$

EXERCISES

Applying Milne's method compute y at the specified value of x for the following. [1 to 4]

1. $\frac{dy}{dx} + xy^2 = 0$

x	0	0.2	0.4	0.6
y	2	1.9231	1.7241	1.4706

Compute $y(0.8)$

2. $\frac{dy}{dx} = 1 + y^2$; $y(0) = 0$. Compute $y(0.8)$ correct to four decimal places by generating the initial values from Picard's second approximation.

3. $\frac{dy}{dx} = x + y^2$

x	0	0.1	0.2	0.3
y	1	1.1	1.231	1.402

Compute $y(0.4)$

4. $\frac{dy}{dx} = \frac{1}{2}(x+y)$, $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$, $y(1.5) = 4.968$. Compute y at $x = 2$ correct to three decimal places.

5. Use Taylor's series method to obtain the solution as a power series in x (upto the third derivative terms) given that $\frac{dy}{dx} + y^2 = x$, $y(0) = 0$. Using this generate the values of y corresponding to $x = 0.2$, 0.4, 0.6 correct to four decimal places. Then apply Milne's predictor-corrector formulae to compute y at $x = 0.8$ and at $x = 1$

6. Using Adams - Bashforth method find $y(1.4)$ given that $\frac{dy}{dx} = x^2 + \frac{y}{2}$ with $y(1) = 2$. Obtain the initial values of y at $x = 1.1, 1.2, 1.3$ by Taylor's series method of order 4.

7. Given $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ and the data :

x	1	1.1	1.2	1.3
y	1	0.996	0.986	0.972

compute $y(1.4)$ correct to 3 decimal places by applying Adams - Bashforth predictor and corrector formulae.

8. Solve $\frac{dy}{dx} + y = x^2$; $y(0) = 1$ by obtaining Taylor's polynomial of order 4. Evaluate y at $x = 0.1$, 0.2, 0.3 and by using these values obtain y at $x = 0.4$ applying Adams - Bashforth predictor and corrector formulae.

ANSWERS

1. 1.22
2. 1.0234
3. 1.7003
4. 6.873
5. 0.3046, 0.4555
6. 3.0793
7. 0.949
8. 0.6897

PART A

Unit - II

Numerical Methods - 2

2.1 Introduction

In this unit we discuss numerical solution of simultaneous first order ODEs and also second order ODEs as an extension of some of the earlier discussed methods (*Unit-I*) for solving ODEs of first order.

2.2 Numerical solution of simultaneous first order ODEs

2.21 Picard's Method

Consider the following system of equations : (*Simultaneous equations*)

$$\frac{dy}{dx} = f(x, y, z) \qquad \ldots (1)$$

and $$\frac{dz}{dx} = g(x, y, z) \qquad \ldots (2)$$

with the initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$

That is, $y = y_0$ and $z = z_0$ at $x = x_0$.

We have to find successive approximations for y and z interms of x.

We have from (1) and (2),

$$dy = f(x, y, z)\,dx \quad ; \quad dz = g(x, y, z)\,dx$$

where, $y = y_0$ at $x = x_0$; $z = z_0$ at $x = x_0$

Integrating LHS between y_0 and y, RHS between x_0 and x we have,

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y, z)\,dx$$

Integrating LHS between z_0 and z, RHS between x_0 and x we have,

$$\int_{z_0}^{z} dz = \int_{x_0}^{x} g(x, y, z)\,dx$$

ie., $y - y_0 = \int_{x_0}^{x} f(x, y, z)\, dx$; $z - z_0 = \int_{x_0}^{x} g(x, y, z)\, dx$

or $y = y_0 + \int_{x_0}^{x} f(x, y, z)\, dx$; $z = z_0 + \int_{x_0}^{x} g(x, y, z)\, dx$

The first approximation for y and z denoted respectively by y_1 and z_1 are obtained by replacing the given initial values $y = y_0$ and $z = z_0$ in the RHS of these expressions. That is ,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0, z_0)\, dx \quad ; \quad z_1 = z_0 + \int_{x_0}^{x} g(x, y_0, z_0)\, dx$$

Similarly we obtain the second approximation y_2 and z_2. That is

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1, z_1)\, dx \quad ; \quad z_2 = z_0 + \int_{x_0}^{x} g(x, y_1, z_1)\, dx \text{ and so on.}$$

WORKED PROBLEMS

1. *Use Picard's method to find* $y(0.1)$ *and* $z(0.1)$ *given that* $\frac{dy}{dx} = x + z$, $\frac{dz}{dx} = x - y^2$ *and* $y(0) = 2$, $z(0) = 1$. *(Carry out two approximations)*

>> We have a system of two equations and we need to find two approximations for y and z as functions of x.

$$\frac{dy}{dx} = x + z \ ; \ y(0) = 2 \quad ; \quad \frac{dz}{dx} = x - y^2 \ ; \ z(0) = 1$$

$$dy = (x + z)\, dx \ : \ y = 2,\ x = 0 \ ; \quad dz = (x - y^2)\, dx \ : \ z = 1,\ x = 0$$

Now, $\int_2^y dy = \int_0^x (x + z)\, dx$; $\int_1^z dz = \int_0^x (x - y^2)\, dx$

Hence,

$$y = 2 + \int_0^x (x + z)\, dx \quad \ldots (1)$$

$$z = 1 + \int_0^x (x - y^2)\, dx \quad \ldots (2)$$

The first approximation for y and z are obtained by replacing the initial values of y and z in the RHS of (1) & (2).

$$y_1 = 2 + \int_0^x (x+1)\,dx \quad : \quad z_1 = 1 + \int_0^x (x - 2^2)\,dx$$

$$\therefore \quad y_1 = 2 + x + \frac{x^2}{2} \quad : \quad z_1 = 1 - 4x + \frac{x^2}{2}$$

Now, $y_2 = 2 + \int_0^x (x + z_1)\,dx \quad : \quad z_2 = 1 + \int_0^x (x - y_1^2)\,dx$

$$= 2 + \int_0^x \left[x + \left(1 - 4x + \frac{x^2}{2}\right)\right] dx \quad : \quad = 1 + \int_0^x \left[x - \left(2 + x + \frac{x^2}{2}\right)^2 \right] dx$$

$$= 2 + \int_0^x \left(1 - 3x + \frac{x^2}{2}\right) dx \quad : \quad = 1 + \int_0^x \left[x - \left(4 + x^2 + \frac{x^4}{4} + 4x + x^3 + 2x^2\right)\right] dx$$

$$= 2 + x - \frac{3x^2}{2} + \frac{x^3}{6} \quad : \quad = 1 + \int_0^x \left(-4 - 3x - 3x^2 - x^3 - \frac{x^4}{4}\right) dx$$

$$\therefore \quad y_2 = 2 + x - \frac{3x^2}{2} + \frac{x^3}{6} \quad : \quad z_2 = 1 - 4x - \frac{3x^2}{2} - x^3 - \frac{x^4}{4} - \frac{x^5}{20}$$

These are the required second approximation for y, z and now putting $x = 0.1$ in these we have

$$y(0.1) = 2 + (0.1) - \frac{3(0.1)^2}{2} + \frac{(0.1)^3}{6} = 2.0852$$

$$z(0.1) = 1 - 4(0.1) - \frac{3(0.1)^2}{2} - (0.1)^3 - \frac{(0.1)^4}{4} - \frac{(0.1)^5}{20} = 0.584$$

Thus **$y(0.1) = 2.0852$ and $z(0.1) = 0.584$**

---

2. *Obtain second approximation to the solution of* $\frac{dx}{dt} = \frac{1}{2}(y+1)$, $\frac{dy}{dt} = \frac{2t}{3x}$ $x = 2/3$, $y = 3$ *at* $t = 1$ *using Picard's method*

>> $\frac{dx}{dt} = \frac{1}{2}(y + 1) \quad ; \quad \frac{dy}{dt} = \frac{2t}{3x}$

$$dx = \frac{1}{2}(y + 1)\,dt : x = 2/3,\ t = 1 \quad ; \quad dy = \frac{2t}{3x}\,dt : y = 3, t = 1$$

Now, $\int_{2/3}^{x} dx = \int_1^t \frac{1}{2}(y+1)\,dt \quad ; \quad \int_3^y dy = \int_1^t \frac{2t}{3x}\,dt$

Hence, $$x = \frac{2}{3} + \frac{1}{2}\int_1^t (y+1)\,dt \qquad \ldots(1)$$

$$y = 3 + \int_1^t \frac{2t}{3x}\,dt \qquad \ldots(2)$$

Putting $y = 3$ in the RHS of (1) and $x = 2/3$ in the RHS of (2) we have,

$$x_1 = \frac{2}{3} + \frac{1}{2}\int_1^t 4\,dt \quad : \quad y_1 = 3 + \int_1^t \frac{2t}{3 \cdot 2/3}\,dt$$

$$= \frac{2}{3} + \Big[2t\Big]_1^t \quad : \quad = 3 + \left[\frac{t^2}{2}\right]_1^t$$

$$= \frac{2}{3} + 2t - 2 \quad : \quad = 3 + \frac{t^2}{2} - \frac{1}{2}$$

$$\therefore \quad x_1 = 2t - \frac{4}{3} \quad : \quad y_1 = \frac{5}{2} + \frac{t^2}{2}$$

Now, $$x_2 = \frac{2}{3} + \frac{1}{2}\int_1^t (y_1 + 1)\,dt \quad : \quad y_2 = 3 + \int_1^t \frac{2t}{3x_1}\,dt$$

$$= \frac{2}{3} + \frac{1}{2}\int_1^t \left(\frac{5}{2} + \frac{t^2}{2} + 1\right) dt \quad : \quad = 3 + \int_1^t \frac{2t}{3\,(2t - 4/3)}\,dt$$

$$= \frac{2}{3} + \frac{1}{2}\int_1^t \left(\frac{7}{2} + \frac{t^2}{2}\right) dt \quad : \quad = 3 + \int_1^t \frac{2t}{6t - 4}\,dt$$

$$= \frac{2}{3} + \frac{1}{2}\left[\frac{7t}{2} + \frac{t^3}{6}\right]_1^t \quad : \quad = 3 + \int_1^t \frac{t}{3t - 2}\,dt$$

$$= \frac{2}{3} + \frac{1}{2}\left[\left(\frac{7t}{2} + \frac{t^3}{6}\right) - \left(\frac{7}{2} + \frac{1}{6}\right)\right] \quad : \quad = 3 + \frac{1}{3}\int_1^t \frac{3t}{3t - 2}\,dt$$

$$x_2 = \frac{2}{3} + \frac{1}{2}\left[\frac{7t}{2} + \frac{t^3}{6} - \frac{11}{3}\right] \quad : \quad y_2 = 3 + \frac{1}{3}\int_1^t \frac{(3t-2) + 2}{3t-2}\,dt$$

$$x_2 = \frac{7t}{4} + \frac{t^3}{12} + \frac{2}{3} - \frac{11}{6} \quad : \quad y_2 = 3 + \frac{1}{3}\int_1^t \left(1 + \frac{2}{3t-2}\right) dt$$

$$= \frac{7t}{4} + \frac{t^3}{12} - \frac{7}{6} \quad : \quad = 3 + \frac{1}{3}\left[t + \frac{2}{3}\log(3t-2)\right]_1^t$$

$$x_2 = \frac{7t}{4} + \frac{t^3}{12} - \frac{7}{6} \quad : \quad = 3 + \frac{1}{3}\left[t + \frac{2}{3}\log(3t-2) - 1\right]$$

$$\therefore \quad x_2 = \frac{7t}{4} + \frac{t^3}{12} - \frac{7}{6} \quad : \quad y_2 = \frac{8}{3} + \frac{t}{3} + \frac{2}{9}\log(3t-2)$$

Thus the required second approximations are

$$x_2 = \frac{7t}{4} + \frac{t^3}{12} - \frac{7}{6} \; ; \; y_2 = \frac{8}{3} + \frac{t}{3} + \frac{2}{9}\log(3t-2)$$

---

3. *Find the second approximation for the solution of the following system of equations by applying Picard's method.* $\frac{dx}{dt} = (x+y)t$, $\frac{dy}{dt} = (x-t)y$; $x = 0$, $y = 1$ *at* $t = 0$.

>> We have by data,

$$dx = (x+y)t\,dt : x = 0,\ t = 0 \quad ; \quad dy = (x-t)y : y = 1,\ t = 0$$

Now, $\int_0^x dx = \int_0^t (x+y)t\,dt \quad ; \quad \int_1^y dy = \int_0^t (x-t)y\,dt$

Hence, $$x = \int_0^t (x+y)t\,dt \qquad \ldots (1)$$

$$y = 1 + \int_0^t (x-t)y\,dt \qquad \ldots (2)$$

Putting $x = 0, y = 1$ in the RHS of (1) and (2) we have,

$$x_1 = \int_0^t t\,dt \quad : \quad y_1 = 1 + \int_0^t -t\,dt$$

$$\therefore \quad x_1 = \frac{t^2}{2} \quad : \quad y_1 = 1 - \frac{t^2}{2}$$

$$x_2 = \int_0^t (x_1 + y_1)\, t\, dt \quad : \quad y_2 = 1 + \int_0^t (x_1 - t)\, y_1\, dt$$

$$x_2 = \int_0^t t\, dt \quad : \quad y_2 = 1 + \int_0^t \left(\frac{t^2}{2} - t\right)\left(1 - \frac{t^2}{2}\right) dt$$

$$x_2 = \frac{t^2}{2} \quad : \quad y_2 = 1 + \int_0^t \left(-t + \frac{t^2}{2} + \frac{t^3}{2} - \frac{t^4}{4}\right) dt$$

Thus $x_2 = \frac{t^2}{2}$, $y_2 = 1 - \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{8} - \frac{t^5}{20}$ are the required second approximations.

---

4. *Evaluate y and z at $x = 0.1$ from the Picard's second approximation to the solution of the following system of equations given that $y = 1$ and $z = 0.5$ at $x = 0$ initially.*

$$\frac{dy}{dx} = z, \quad \frac{dz}{dx} = x^3 (y + z)$$

>> We have by data,

$$dy = z\, dx \; ; \; y = 1, \; x = 0 \quad ; \quad dz = x^3 (y + z)\, dx \; ; \; z = 1/2, \; x = 0$$

Now, $\int_1^y dy = \int_0^x z\, dx \quad ; \quad \int_{1/2}^z dz = \int_0^x x^3 (y + z)\, dx$

Hence, $$y = 1 + \int_0^x z\, dx \qquad \ldots (1)$$

$$z = \frac{1}{2} + \int_0^x x^3 (y + z)\, dx \qquad \ldots (2)$$

Putting $z = 1/2$ in the RHS of (1) and $y = 1, z = 1/2$ in the RHS of (2), we have,

$$y_1 = 1 + \int_0^x \frac{1}{2}\, dx \quad : \quad z_1 = \frac{1}{2} + \int_0^x \frac{3}{2} x^3\, dx$$

$$\therefore \quad y_1 = 1 + \frac{x}{2} \quad : \quad z_1 = \frac{1}{2} + \frac{3x^4}{8}$$

Now, $y_2 = 1 + \int_0^x z_1\, dx \quad : \quad z_2 = \frac{1}{2} + \int_0^x x^3 (y_1 + z_1)\, dx$

$$y_2 = 1 + \int_0^x \left(\frac{1}{2} + \frac{3x^4}{8}\right) dx \quad : \quad z_2 = \frac{1}{2} + \int_0^x x^3 \left(\frac{3}{2} + \frac{x}{2} + \frac{3x^4}{8}\right) dx$$

$$\therefore \quad y_2 = 1 + \frac{x}{2} + \frac{3x^5}{40} \quad : \quad z_2 = \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}$$

Thus by putting $x = 0.1$ in the RHS of these expressions we obtain,

$y(0.1) = \mathbf{1.05000075}$ and $z(0.1) = \mathbf{0.5000385}$ or

$y(0.1) \approx \mathbf{1.05}$ **and** $z(0.1) \approx \mathbf{0.5}$

---

5. *Obtain the Picard's third approximation to the solution of the system of equations* $\dfrac{dx}{dt} = 2x + 3y$, $\dfrac{dy}{dt} = x - 3y$; $t = 0$, $x = 0$, $y = 1/2$. *Hence find* x *and* y *at* $t = 0.2$

>> We have by data,

$$dx = (2x + 3y)\, dt \;:\; x = 0, t = 0 \;:\; dy = (x - 3y)\, dt \;:\; y = 1/2,\ t = 0$$

Now $\displaystyle\int_0^x dx = \int_0^t (2x + 3y)\, dt \quad : \quad \int_{1/2}^y dy = \int_0^t (x - 3y)\, dt$

Hence, $\displaystyle x = \int_0^t (2x + 3y)\, dt$ ... (1)

$$y = \frac{1}{2} + \int_0^t (x - 3y)\, dt \qquad \ldots (2)$$

Putting $x = 0, y = 1/2$ in the RHS of (1) and (2) we have,

$$x_1 = \int_0^t \frac{3}{2}\, dt \quad : \quad y_1 = \frac{1}{2} + \int_0^t -\frac{3}{2}\, dt$$

$$\therefore \quad x_1 = \frac{3t}{2} \quad : \quad y_1 = \frac{1}{2} - \frac{3t}{2}$$

Now, $\displaystyle x_2 = \int_0^t (2x_1 + 3y_1)\, dt \quad : \quad y_2 = \frac{1}{2} + \int_0^t (x_1 - 3y_1)\, dt$

$$x_2 = \int_0^t \left(\frac{3}{2} - \frac{3t}{2}\right) dt \quad : \quad y_2 = \frac{1}{2} + \int_0^t \left(-\frac{3}{2} + 6t\right) dt$$

$\therefore \quad x_2 = \dfrac{3t}{2} - \dfrac{3t^2}{4} \quad : \quad y_2 = \dfrac{1}{2} - \dfrac{3t}{2} + 3t^2$

Now, $x_3 = \int_0^t (2x_2 + 3y_2)\, dt \quad : \quad y_3 = \dfrac{1}{2} + \int_0^t (x_2 - 3y_2)\, dt$

$$x_3 = \int_0^t \left(\frac{3}{2} - \frac{3t}{2} + \frac{15t^2}{2} \right) dt \quad : \quad y_3 = \frac{1}{2} + \int_0^t \left(-\frac{3}{2} + 6t - \frac{39t^2}{4} \right) dt$$

$\therefore \quad x_3 = \dfrac{3t}{2} - \dfrac{3t^2}{4} + \dfrac{5t^3}{2} \quad : \quad y_3 = \dfrac{1}{2} - \dfrac{3t}{2} + 3t^2 - \dfrac{39t^3}{12}$

Thus by putting $t = 0.2$ in these we obtain $x(0.2) = 0.29$ and $y(0.2) = 0.294$

2.22 Runge-Kutta method of fourth order

Consider the system of equations

$$\frac{dy}{dx} = f(x, y, z) \quad \ldots (1)$$

and $$\frac{dz}{dx} = g(x, y, z) \quad \ldots (2)$$

with the initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$.

That is, $y = y_0$, $z = z_0$ at $x = x_0$

We have to compute $y(x_0 + h)$ and $z(x_0 + h)$

We need to first compute the following :

$$k_1 = hf(x_0, y_0, z_0) \quad ; \quad l_1 = hg(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2} \right) \quad ; \quad l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2} \right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2} \right) \quad ; \quad l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2} \right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) \quad ; \quad l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$$

The required $y(x_0 + h) = y_0 + \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

and $z(x_0 + h) = z_0 + \dfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

Note : The values of l_1, l_2, l_3, l_4 *is got by applying the definition of* $g(x, y, z)$.

We need to compute the pair of values in the following order.

(k_1, l_1); (k_2, l_2); (k_3, l_3); (k_4, l_4) *and finally compute* $y(x_0+h)$ *and* $z(x_0+h)$

---

6. *Use fourth order Runge-Kutta method to solve the system of equations :*
$\frac{dx}{dt} = y - t$, $\frac{dy}{dt} = x + t$, $x = 1, y = 1$ *at* $t = 0$. *Compute* $x(0.1)$ *and* $y(0.1)$.

>> Let $f(t, x, y) = y - t$, $g(t, x, y) = x + t$; $t_0 = 0$, $x_0 = 1$, $y_0 = 1$

Let us take $h = 0.1$ since we need to compute $x(0.1)$ and $y(0.1)$.

We shall first compute the following.

$$k_1 = hf(t_0, x_0, y_0) = (0.1)f(0, 1, 1) = (0.1)[1 - 0] = 0.1$$

$$l_1 = hg(t_0, x_0, y_0) = (0.1)g(0, 1, 1) = (0.1)[1 + 0] = 0.1$$

$$k_2 = hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}, y_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.1)f(0.05, 1.05, 1.05) = (0.1)[1.05 - 0.05] = 0.1$$

$$l_2 = hg\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}, y_0 + \frac{l_1}{2}\right)$$

$$l_2 = (0.1)g(0.05, 1.05, 1.05) = (0.1)[1.05 + 0.05] = 0.11$$

$$k_3 = hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}, y_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.1)f(0.05, 1.05, 1.055) = (0.1)[1.055 - 0.05] = 0.1005$$

$$l_3 = hg\left(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}, y_0 + \frac{l_2}{2}\right)$$

$$l_3 = (0.1)g(0.05, 1.05, 1.055) = (0.1)[1.05 + 0.05] = 0.11$$

$$k_4 = hf(t_0 + h, x_0 + k_3, y_0 + l_3)$$

$$k_4 = (0.1)f(0.1, 1.1005, 1.11) = (0.1)[1.11 - 0.1] = 0.101$$

$$l_4 = hg(t_0 + h, x_0 + k_3, y_0 + l_3)$$

$$l_4 = (0.1)g(0.1, 1.1005, 1.11) = (0.1)[1.1005 + 0.1] = 0.12005$$

We have, $x(t_0 + h) = x_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$y(t_0 + h) = y_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$\therefore$ $x(0.1) = 1 + \frac{1}{6}[0.1 + 2(0.1) + 2(0.1005) + 0.101]$

$$y(0.1) = 1 + \frac{1}{6}[0.1 + 2(0.11) + 2(0.11) + 0.12005]$$

Thus **$x(0.1) = 1.100333 \approx 1.1$; $y(0.1) = 1.1100083 \approx 1.11$**

Note : *The values of l_1, l_2, l_3, l_4 is got simply by applying the definition of $g(t, x, y)$ In the subsequent problems we do not give the formula associated with "g"*

---

7. *Solve* $\frac{dy}{dx} = 1 + zx$, $\frac{dz}{dx} + xy = 0$, $y(0) = 0$, $z(0) = 1$ *at* $x = 0.3$ *by applying fourth order Runge-Kutta method .*

>> Let $f(x, y, z) = 1 + zx$, $g(x, y, z) = -xy$; $x_0 = 0$, $y_0 = 0$, $z_0 = 1$.

Let us choose $h = 0.3$ and we shall first compute the following :

$$k_1 = hf(x_0, y_0, z_0)$$

$$k_1 = (0.3) f(0, 0, 1) = (0.3)[1 + 1 \times 0] = 0.3$$

$l_1 = (0.3)(-0 \times 0) = 0$, by applying the definition of 'g'

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.3) f(0.15, 0.15, 1) = (0.3)[1 + (1)(0.15)] = 0.345$$

$$l_2 = (0.3)[-(0.15)(0.15)] = -0.00675$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.3) f(0.15, 0.1725, 0.996625) = (0.3)[1 + (0.996625)(0.15)] = 0.34485$$

$$l_3 = (0.3)[-(0.15)(0.1725)] = -0.0077625$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k_4 = (0.3) f(0.3, 0.34485, 0.99224) = (0.3)[1 + (0.99224)(0.3)] = 0.3893$$

$$l_4 = (0.3)[-(0.3)(0.34485)] = -0.03104$$

We have, $y(x_0+h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$z(x_0+h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$\therefore \quad y(0.3) = 0 + \frac{1}{6}\left[0.3 + 2(0.345) + 2(0.34485) + 0.3893\right]$$

$$z(0.3) = 1 + \frac{1}{6}\left[0 + 2(-0.00675) + 2(-0.0077625) + (-0.03104)\right]$$

Thus **$y(0.3) = 0.34483$ and $z(0.3) = 0.98999$**

---

8. *Solve* $\frac{dy_1}{dx} = y_1 + y_2$, $\frac{dy_2}{dx} = -y_1 + y_2$ *given* $y_1(0) = 0$, $y_2(0) = 1$ *at* $x = 0.1$ *by taking* $h = 0.1$ *applying fourth order Runge-Kutta method.*

>> Let $f(x, y_1, y_2) = y_1 + y_2$, $g(x, y_1, y_2) = y_2 - y_1$

$$x_0 = 0,\ (y_1)_0 = 0,\ (y_2)_0 = 1,\ h = 0.1$$

We shall first compute the following.

$$k_1 = hf(x_0, (y_1)_0, (y_2)_0)$$

$$k_1 = (0.1)f(0, 0, 1) = (0.1)[0+1] = 0.1$$

$l_1 = (0.1)[1-0] = 0.1$, by applying the definition of 'g'

$$k_2 = hf\left(x_0 + \frac{h}{2}, (y_1)_0 + \frac{k_1}{2}, (y_2)_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.1)f(0.05, 0.05, 1.05) = (0.1)[0.05 + 1.05] = 0.11$$

$$l_2 = (0.1)[1.05 - 0.05] = 0.1$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, (y_1)_0 + \frac{k_2}{2}, (y_2)_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.1)f(0.05, 0.055, 1.05) = (0.1)[0.055 + 1.05] = 0.1105$$

$$l_3 = (0.1)[1.05 - 0.055] = 0.0995$$

$$k_4 = hf(x_0 + h, (y_1)_0 + k_3, (y_2)_0 + l_3)$$

$$k_4 = (0.1)f(0.1, 0.1105, 1.0995) = (0.1)[0.1105 + 1.0995] = 0.121$$

$$l_4 = (0.1)[1.0995 - 0.1105] = 0.0989$$

We have, $y_1(x_0+h) = (y_1)_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$y_2(x_0+h) = (y_2)_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$\therefore \quad y_1(0.1) = 0 + \frac{1}{6}\left[0.1 + 2(0.11) + 2(0.1105) + 0.121\right]$$

$$y_2(0.1) = 1 + \frac{1}{6}\left[0.1 + 2(0.1) + 2(0.0995) + 0.0989\right]$$

Thus $\mathbf{y_1(0.1) = 0.11033}$ **and** $\mathbf{y_2(0.1) = 1.09965}$

---

9. *Given* $\frac{dv}{du} = uvw$, $\frac{dw}{du} = \frac{uv}{w}$, $v(1) = 1/3$, $w(1) = 1$, *compute* $v(1.1)$ *and* $w(1.1)$ *by applying Runge-Kutta method of order-4.*

>> Let $f(u, v, w) = uvw$, $g(u, v, w) = \frac{uv}{w}$; $u_0 = 1$, $v_0 = 1/3$, $w_0 = 1$.
Let us choose $h = 0.1$ and we shall first compute the following.

$$k_1 = hf(u_0, v_0, w_0)$$

$$k_1 = (0.1)f(1, 1/3, 1) = (0.1)[(1)(1/3)(1)] = 0.0333$$

$$l_1 = (0.1)\left[\frac{(1)(1/3)}{1}\right] = 0.0333, \text{ by applying the definition of 'g'}$$

$$k_2 = hf\left(u_0 + \frac{h}{2}, v_0 + \frac{k_1}{2}, w_0 + \frac{l_1}{2}\right)$$

$$k_2 = 0.1\, f(1.05, 0.35, 1.01665) = 0.03736$$

$$l_2 = (0.1)\left[\frac{(1.05)(0.35)}{1.01665}\right) = 0.03615$$

$$k_3 = (0.1)f\left(u_0 + \frac{h}{2}, v_0 + \frac{k_2}{2}, w_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.1)f(1.05, 0.352, 1.0181) = 0.03763$$

$$l_3 = (0.1)\left[\frac{(1.05)(0.352)}{1.0181}\right] = 0.0363$$

$$k_4 = (0.1) f(u_0 + h, v_0 + k_3, w_0 + l_3)$$

$$k_4 = (0.1) f(1.1, 0.371, 1.0363) = 0.0423$$

$$l_4 = (0.1)\left[\frac{(1.1)(0.371)}{1.0363}\right] = 0.03938$$

We have, $v(u_0 + h) = v_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$w(u_0 + h) = w_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

Thus by substituting the appropriate values we obtain

$$\mathbf{v(1.1) = 0.3709 \text{ and } w(1.1) = 1.03626}$$

---

10. *Solve the following system of equations to find the values of x and y at* $t = 0.2$ *given* $\frac{dx}{dt} = 2x + y$, $\frac{dy}{dt} = x - 3y$ *and* $(t_0, x_0, y_0) = (0, 0, 0.5)$ *using Runge-Kutta method of order - 4.*

>> Let $f(t, x, y) = 2x + y$, $g(t, x, y) = x - 3y$; $t_0 = 0, x_0 = 0, y_0 = 0.5$.

Let us take $h = 0.2$ and we shall first compute the following :

$$k_1 = hf(t_0, x_0, y_0)$$

$$k_1 = (0.2) f(0, 0, 0.5) = (0.2)[2(0) + 0.5] = 0.1$$

$l_1 = (0.2)[0 - 3(0.5)] = -0.3$ by applying the definition of 'g'

$$k_2 = (0.2) f\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}, y_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.2) f(0.1, 0.05, 0.35) = (0.2)[2(0.05) + 0.35] = 0.09$$

$$l_2 = (0.2)[0.05 - 3(0.35)] = -0.2$$

$$k_3 = (0.2) f\left(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}, y_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.2) f(0.1, 0.045, 0.4) = (0.2)[2(0.045) + 0.4] = 0.098$$

$$l_3 = (0.2)[0.045 - 3(0.4)] = -0.231$$

$$k_4 = (0.2) f(t_0 + h, x_0 + k_3, y_0 + l_3)$$

$$k_4 = (0.2) f(0.2, 0.098, 0.269) = (0.2)[2(0.098) + 0.269] = 0.093$$

$l_4 = (0.2)[0.098 - 3(0.269)] = -0.142$

We have $x(t_0 + h) = x_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$y(t_0 + h) = y_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

Thus by substituting the appropriate values we obtain,

$$\mathbf{x(0.2) = 0.0948, \quad y(0.2) = 0.2827}$$

2.3 Numerical solution of second order ODEs by Picard's method and Runge-Kutta method.

The given second order ODE with two initial conditions will reduce to two first order simultaneous ODEs which can be solved by both the methods discussed in the previous article-***Picards method*** and ***Runge-Kutta method*** of fourth order.

We present the method explicitly.

Let $y'' = g(x, y, y')$ with the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$ be the given second order d.e.

Now, let $y' = \frac{dy}{dx} = z$. This gives $y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx}$

The given second order d.e. assumes the form : $\frac{dz}{dx} = g(x, y, z)$ with the conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ where y_0' is be denoted by z_0.

Hence we now have two first order simultaneous ODEs,

$$\frac{dy}{dx} = z \text{ and } \frac{dz}{dx} = g(x, y, z) \text{ with } y(x_0) = y_0 \text{ and } z(x_0) = z_0$$

Taking $f(x, y, z) = z$, we now have the following system of equations for solving.

$$\frac{dy}{dx} = f(x, y, z), \quad \frac{dz}{dx} = g(x, y, z) \ ; \quad y(x_0) = y_0 \text{ and } z(x_0) = z_0$$

The method of solving these by Picard's method is described in the article 2.21 and by Runge-Kutta method of fourth order is described in the article 2.22.

WORKED PROBLEMS

11. *Use Picard's method to obtain the third approximation to the solution of* $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 6y = 0$ *given that when* $x = 0,\ y = 1,\ \frac{dy}{dx} = 0.1$

>> The given equation is a second order differential equation and we convert this into two first order simultaneous equations using a substitution and employ Picard's method to find the third approximation for y.

Put $\frac{dy}{dx} = z$. Differentiating *w.r.t.* x we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$ so that the given equation becomes $\frac{dz}{dx} + 3xz - 6y = 0$

Now we have the system of equations,

$\frac{dy}{dx} = z\ :\ y = 1,\ x = 0$: $\frac{dz}{dx} + 3xz - 6y = 0\ :\ z = 0.1, x = 0$

$dy = z\,dx$: $dz = (6y - 3xz)\,dx$

Integrating LHS between 1 & y, RHS between 0 & x, we have, : Integrating LHS between 0.1 & z, RHS between 0 & x, we have

$\int_1^y dy = \int_0^x z\,dx$: $\int_{0.1}^z dz = \int_0^x (6y - 3xz)\,dx$

$y - 1 = \int_0^x z\,dx$: $z - 0.1 = \int_0^x (6y - 3xz)\,dx$

Hence, $y = 1 + \int_0^x z\,dx \quad \ldots(1)$: $z = \frac{1}{10} + \int_0^x (6y - 3xz)\,dx \quad \ldots(2)$

The first approximation for y is obtained by putting $z = 0.1$ in the RHS of (1). : The first approximation for z is obtained by putting $y = 1,\ z = 0.1$ in the RHS of (2).

$y_1 = 1 + \int_0^x 0.1\,dx$: $z_1 = \frac{1}{10} + \int_0^x \left(6 - \frac{3x}{10}\right) dx$

$\therefore \quad y_1 = 1 + (0.1)x = 1 + \frac{x}{10} \quad : \quad z_1 = \frac{1}{10} + 6x - \frac{3x^2}{20}$

$y_2 = 1 + \int_0^x z_1\, dx \quad : \quad z_2 = \frac{1}{10} + \int_0^x (6y_1 - 3xz_1)\, dx$

$y_2 = 1 + \int_0^x \left(\frac{1}{10} + 6x - \frac{3x^2}{20} \right) dx \quad : \quad z_2 = \frac{1}{10} + \int_0^x \left[6\left(1 + \frac{x}{10}\right) - 3x\left(\frac{1}{10} + 6x - \frac{3x^2}{20}\right) \right] dx$

$\therefore \quad y_2 = 1 + \frac{x}{10} + 3x^2 - \frac{x^3}{20} \quad : \quad z_2 = \frac{1}{10} + \int_0^x \left(6 + \frac{3x}{10} - 18x^2 + \frac{9x^3}{20} \right) dx$

$y_3 = 1 + \int_0^x z_2\, dx \quad : \quad z_2 = \frac{1}{10} + 6x + \frac{3x^2}{20} - 6x^3 + \frac{9x^4}{80}$

z_3 is not required

$$= 1 + \int_0^x \left(\frac{1}{10} + 6x + \frac{3x^2}{20} - 6x^3 + \frac{9x^4}{80} \right) dx$$

Thus $y = 1 + \frac{x}{10} + 3x^2 + \frac{x^3}{20} - \frac{3x^4}{2} + \frac{9x^5}{400}$ **is the required third approximation.**

---

12. *Use Picard's method to obtain the second approximation to the solution of* $\frac{d^2y}{dx^2} - x^3 \frac{dy}{dx} - x^3 y = 0$ *given* $y(0) = 1,\ y'(0) = 1/2$ *and hence find* $y(0.1)$.

>> Putting $\frac{dy}{dx} = z$ and differentiating *w.r.t* x we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$

The given equation becomes

$$\frac{dz}{dx} - x^3 z - x^3 y = 0 \ ; \quad y(0) = 1,\ z(0) = 1/2$$

Now we have the system of equations,

$\frac{dy}{dx} = z : y = 1, \ x = 0$: $\frac{dz}{dx} = x^3(y+z) : z = 1/2, \ x = 0$

or $dy = z\,dx$: $dz = x^3 (y+z)\,dx$

Integrating LHS between 1 & y, RHS between 0 & x we have, : Integrating LHS between 1/2 & z, RHS between 0 & x we have,

$\int_1^y dy = \int_0^x z\,dx$: $\int_{1/2}^z dz = \int_0^x x^3(y+z)\,dx$

Hence, $y = 1 + \int_0^x z\,dx$ ...(1) : $z = \frac{1}{2} + \int_0^x x^3(y+z)\,dx$ ...(2)

Putting $z = 1/2$ in the RHS of (1), : Putting $y = 1, z = 1/2$ in the RHS of (2),

$y_1 = 1 + \int_0^x \frac{1}{2}\,dx$: $z_1 = \frac{1}{2} + \int_0^x x^3\left(1 + \frac{1}{2}\right) dx$

$\therefore \ y_1 = 1 + \frac{x}{2}$: $z_1 = \frac{1}{2} + \int_0^x \frac{3}{2} x^3\,dx$

$y_2 = 1 + \int_0^x z_1\,dx$: $z_1 = \frac{1}{2} + \frac{3x^4}{8}$

$y_2 = 1 + \int_0^x \left(\frac{1}{2} + \frac{3x^4}{8}\right) dx$: **z_2 is not required**

$\therefore \ y_2 = 1 + \frac{x}{2} + \frac{3x^5}{40}$

Thus $y = 1 + \frac{x}{2} + \frac{3x^5}{40}$ is the required second approximation.

Now by putting $x = 0.1$ we obtain $y(0.1) = 1.0500008$

---

13. *Use Picard's method to find the third approximation to the solution of the equation* $\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + y = 0$ *given that* $y = 0.5$, $\dfrac{dy}{dx} = 0.1$ *when* $x = 0$. *Also tabulate the value of* $y\,(0.1)$ *from all the three approximations.*

>> Putting $\dfrac{dy}{dx} = z$ and differentiating *w.r.t* x we obtain $\dfrac{d^2y}{dx^2} = \dfrac{dz}{dx}$

The given equation becomes $\dfrac{dz}{dx} + 2xz + y = 0$; $y(0) = 0.5$, $z(0) = 0.1$

Hence we have the system of equations,

$$\frac{dy}{dx} = z \;:\; y = 1/2,\; x = 0 \quad ; \quad \frac{dz}{dx} = -(2xz + y) \;:\; z = 1/10, x = 0$$

or $\quad dy = z\,dx \quad ; \quad dz = -(2xz + y)\,dx$

Integrating LHS between 1/2 and y ; Integrating LHS between 1/10 and z,
RHS between 0 and x we have, RHS between 0 and x we have,

$$\int_{1/2}^{y} dy = \int_0^x z\,dx \quad ; \quad \int_{1/10}^{z} dz = -\int_0^x (2xz + y)\,dx$$

$$\text{Hence, } y = \frac{1}{2} + \int_0^x z\,dx \quad \ldots (1) \quad ; \quad z = \frac{1}{10} - \int_0^x (2xz + y)\,dx \quad \ldots (2)$$

$$\text{Now, } y_1 = \frac{1}{2} + \int_0^x \frac{1}{10}\,dx \quad ; \quad z = \frac{1}{10} - \int_0^x \left(\frac{x}{5} + \frac{1}{2}\right) dx$$

$$\therefore \quad y_1 = \frac{1}{2} + \frac{x}{10} \quad ; \quad z_1 = \frac{1}{10} - \frac{x}{2} - \frac{x^2}{10}$$

$$y_2 = \frac{1}{2} + \int_0^x z_1\,dx \quad ; \quad z_2 = \frac{1}{10} - \int_0^x (2xz_1 + y_1)\,dx$$

$$y_2 = \frac{1}{2} + \int_0^x \left(\frac{1}{10} - \frac{x}{2} - \frac{x^2}{10}\right) dx \quad ; \quad z_2 = \frac{1}{10} - \int_0^x \left(\frac{x}{5} - x^2 - \frac{x^3}{5} + \frac{1}{2} + \frac{x}{10}\right) dx$$

$$\therefore \quad y_2 = \frac{1}{2} + \frac{x}{10} - \frac{x^2}{4} - \frac{x^3}{30} \quad ; \qquad z_2 = \frac{1}{10} - \frac{x}{2} - \frac{3x^2}{20} + \frac{x^3}{3} + \frac{x^4}{20}$$

$$y_3 = \frac{1}{2} + \int_0^x z_2 \, dx \qquad ; \ z_3 \text{ is not required.}$$

$$= \frac{1}{2} + \int_0^x \left(\frac{1}{10} - \frac{x}{2} - \frac{3x^2}{20} + \frac{x^3}{3} + \frac{x^4}{20} \right) dx$$

Thus $\quad y_3 = \frac{1}{2} + \frac{x}{10} - \frac{x^2}{4} - \frac{x^3}{20} + \frac{x^4}{12} + \frac{x^5}{100}$

The value of y (0.1) from the three approximations are tabulated.

y_1 (0.1)	y_2 (0.1)	y_3 (0.1)
0.51	0.507467	0.507458

14. *When a pendulam swings in a resisting medium, its equation of motion is of the form* $\frac{d^2\theta}{dt^2} + a\frac{d\theta}{dt} + b\sin\theta = 0$, *where a and b are constants. Assuming* $a = 0.2$, $b = 10$ *find the second approximation to the solution of the equation with the initial conditions* $\theta = 0.3$ *radian and* $\frac{d\theta}{dt} = 0$ *when* $t = 0$ *by Picard's method. Also find* θ *at* $t = 0.01$ (0.01) 0.03

>> $\frac{d^2\theta}{dt^2} + a\frac{d\theta}{dt} + b\sin\theta = 0 \quad ; \quad a = 0.2, b = 10$ by data.

Putting $\phi = \frac{d\theta}{dt}$ and differentiating *w.r.t t* will give us $\frac{d\phi}{dt} = \frac{d^2\theta}{dt^2}$

The given equation becomes

$$\frac{d\phi}{dt} + 0.2\phi + 10\sin\theta = 0 \qquad ; \qquad \theta = 0.3, \quad \phi = 0, \quad t = 0.$$

Hence we have the system of equations,

$$\frac{d\theta}{dt} = \phi : \theta = 0.3, \ t = 0 \qquad ; \qquad \frac{d\phi}{dt} = -(0.2\phi + 10\sin\theta) : \phi = 0, t = 0$$

or $\quad d\theta = \phi \, dt \qquad ; \qquad d\phi = -(0.2\phi + 10\sin\theta)\, dt$

Integrating LHS between 0.3 and θ, RHS between 0 and t we have, ; Integrating LHS between 0 and ϕ RHS between 0 and t we have,

$$\int_{0.3}^{\theta} d\theta = \int_{0}^{t} \phi \, dt \quad : \quad \int_{0}^{\phi} d\phi = -\int_{0}^{t} (0.2\phi + 10 \sin\theta)\, dt$$

Hence, $\theta = 0.3 + \int_{0}^{t} \phi \, dt$: $\phi = -\int_{0}^{t} (0.2\phi + 10 \sin\theta)\, dt$

Now, $\theta_1 = 0.3 + \int_{0}^{t} 0 \, dt$: $\phi_1 = -\int_{0}^{t} 10 \sin(0.3)\, dt$

$\therefore \quad \theta_1 = 0.3$: $\phi_1 = -\int_{0}^{t} 2.9552 \, dt = -2.9552\, t$

$\theta_2 = 0.3 + \int_{0}^{t} \phi_1 \, dt$: $\phi_1 = -2.9552\, t$

$\theta_2 = 0.3 - \int_{0}^{t} 2.9552\, t \, dt$: ϕ_2 is not required.

$\therefore \quad \theta_2 = 0.3 - 2.9552 \dfrac{t^2}{2}$

Thus the required second approximation to the solution of the given equation is

$$\boldsymbol{\theta = 0.3 - 1.4776\, t^2}$$

Further **$\theta(0.01) = 0.29985$, $\theta(0.02) = 0.29941$ and $\theta(0.03) = 0.29867$**

---

15. *Apply Picard's method to compute $y(1.1)$ from the second approximation to the solution of the equation $y'' + y^2 y' = x^3$ given that y and y' have the value 1 at $x = 1$.*

\>> Putting $y' = \dfrac{dy}{dx} = z$, we obtain $y'' = \dfrac{d^2 y}{dx^2} = \dfrac{dz}{dx}$

The given equation becomes

$$\frac{dz}{dx} + y^2 z = x^3 \quad ; \quad y(1) = 1, z(1) = 1$$

Hence we have the system of equations,

$\dfrac{dy}{dx} = z : y = 1, x = 1$; $\dfrac{dz}{dx} = x^3 - y^2 z$: $z = 1, x = 1$

or $\quad dy = z\, dx$; $dz = (x^3 - y^2 z)\, dx$

Integrating LHS between 1 and y, RHS between 1 and x, we have ; Integrating LHS between 1 and z, RHS between 1 and x, we have

$$\int_1^y dy = \int_1^x z\,dx \quad ; \quad \int_1^z dz = \int_1^x (x^3 - y^2 z)\,dx$$

Hence, $y = 1 + \int_1^x z\,dx \quad ; \quad z = 1 + \int_1^x (x^3 - y^2 z)\,dx$

Now, $y_1 = 1 + \int_1^x 1\,dx \quad ; \quad z_1 = 1 + \int_1^x (x^3 - 1)\,dx$

$$y_1 = 1 + [x]_1^x \quad ; \quad z_1 = 1 + \left[-x + \frac{x^4}{4}\right]_1^x$$

$$= 1 + (x - 1) \quad ; \quad z_1 = 1 + \left(-x + \frac{x^4}{4}\right) - \left(-1 + \frac{1}{4}\right)$$

$\therefore \quad y_1 = x \quad ; \quad z_1 = \frac{7}{4} - x + \frac{x^4}{4}$

$y_2 = 1 + \int_1^x z_1\,dx \quad ;$ z_2 is not required.

$$= 1 + \int_1^x \left(\frac{7}{4} - x + \frac{x^4}{4}\right) dx$$

$$= 1 + \left[\frac{7x}{4} - \frac{x^2}{2} + \frac{x^5}{20}\right]_1^x = 1 + \left(\frac{7x}{4} - \frac{x^2}{2} + \frac{x^5}{20}\right) - \left(\frac{7}{4} - \frac{1}{2} + \frac{1}{20}\right)$$

$\therefore \quad y_2 = -\frac{3}{10} + \frac{7x}{4} - \frac{x^2}{2} + \frac{x^5}{20}$

Thus by putting $x = 1.1$ in the second approximation we obtain, $\mathbf{y(1.1) = 1.1005255}$

---

16. *Given* $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1,\ y(0) = 1,\ y'(0) = 0$. *Evaluate* $y(0.1)$ *using Runge-Kutta method of order* 4.

>> By data, $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1;\ y = 1,\ y' = 0$ at $x = 0$.

Putting $\frac{dy}{dx} = z$ and differentiating w.r.t. x we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$ so that the given equation assumes the form : $\frac{dz}{dx} - x^2 z - 2xy = 1$

Hence, we have a system of equations,

$$\frac{dy}{dx} = z \; ; \; \frac{dz}{dx} = 1 + 2xy + x^2 z \text{ where } y = 1, \; z = 0, \; x = 0$$

Let $f(x, y, z) = z$, $g(x, y, z) = 1 + 2xy + x^2 z$

$x_0 = 0$, $y_0 = 1$, $z_0 = 0$ and let us take $h = 0.1$

We shall first compute the following :

$$k_1 = hf(x_0, y_0, z_0) = (0.1) f(0, 1, 0) = (0.1)(0) = 0$$

$$l_1 = (0.1)[1 + (2)(0)(1) + (0^2)(0)] = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.1) f(0.05, 1, 0.05) = (0.1)(0.05) = 0.005$$

$$l_2 = (0.1)[1 + (2)(0.05)(1) + (0.05)^2(0.05)] = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.1) f(0.05, 1.0025, 0.055) = (0.1)(0.055) = 0.0055$$

$$l_3 = (0.1)[1 + (2)(0.05)(1.0025) + (0.05)^2(0.055)] = 0.11004$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k_4 = (0.1) f(0.1, 1.0055, 0.11004) = (0.1)(0.11004) = 0.011$$

$$l_4 = (0.1)[1 + (2)(0.1)(1.0055) + (0.1)^2(0.11004)] = 0.12022$$

We have, $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$\therefore \quad y(0.1) = 1 + \frac{1}{6}\left[0 + 2(0.005) + 2(0.0055) + 0.011\right]$$

Thus, $\mathbf{y(0.1) = 1.0053}$

---

17. *By Runge-Kutta method, solve* $\dfrac{d^2y}{dx^2} = x\left(\dfrac{dy}{dx}\right)^2 - y^2$ *for* $x = 0.2$ *correct to four decimal places, using the initial conditions* $y = 1$ *and* $y' = 0$ *when* $x = 0$.

>> By data, $\dfrac{d^2y}{dx^2} = x\left(\dfrac{dy}{dx}\right)^2 - y^2$

Putting $\dfrac{dy}{dx} = z$ and differentiating w.r.t. x, we obtain $\dfrac{d^2y}{dx^2} = \dfrac{dz}{dx}$

The given equation becomes,

$$\frac{dz}{dx} = xz^2 - y^2 \text{ with } y = 1, z = 0 \text{ when } x = 0.$$

Hence, we have a system of equations $\dfrac{dy}{dx} = z,\ \dfrac{dz}{dx} = xz^2 - y^2$

Let $f(x, y, z) = z,\ g(x, y, z) = xz^2 - y^2,\ x_0 = 0,\ y_0 = 1,\ z_0 = 0$ and $h = 0.2$

We shall first compute the following.

$$k_1 = hf(x_0, y_0, z_0) = (0.2)f(0, 1, 0) = (0.2)(0) = 0$$

$$l_1 = (0.2)[(0)(0)^2 - (1)^2] = -0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2},\ z_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.2)f(0.1, 1, -0.1) = (0.2)(-0.1) = -0.02$$

$$l_2 = (0.2)[(0.1)(-0.1)^2 - (1)^2] = -0.1998$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.2)f(0.1, 0.99, -0.0999) = (0.2)(-0.0999) = -0.01998$$

$$l_3 = (0.2)[(0.1)(-0.0999)^2 - (0.99)^2] = -0.1958$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)$$

$$k_4 = (0.2)f(0.2, 0.98002, -0.1958) = (0.2)(-0.1958) = -0.03916$$

$$l_4 = (0.2)[(0.2)(-0.1958)^2 - (0.98002)^2] = -0.19055$$

We have $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$\therefore \quad y(0.2) = 1 + \frac{1}{6}\left[0 + 2(-0.02) + 2(-0.01998) - 0.03916\right]$$

Thus $y(0.2) = \mathbf{0.9801}$

---

18. *Compute $y(0.1)$ given $\frac{d^2y}{dx^2} = y^3$ and $y = 10$, $\frac{dy}{dx} = 5$ at $x = 0$ by Runge-Kutta method of fourth order.*

>> Putting $\frac{dy}{dx} = z$ and differentiating w.r.t x we obtain $\frac{d^2y}{dx^2} = \frac{dz}{dx}$ so that the given equation assumes the form $\frac{dz}{dx} = y^3$. Hence we have a system of equations :

$$\frac{dy}{dx} = z \; ; \; \frac{dz}{dx} = y^3 \text{ where } y = 10, \; z = 5, \; x = 0.$$

Let $f(x, y, z) = z$, $g(x, y, z) = y^3$, $x_0 = 0$, $y_0 = 10$, $z_0 = 5$ and $h = 0.1$

We shall first compute the following.

$$k_1 = hf(x_0, y_0, z_0) = (0.1)f(0, 10, 5) = (0.1)5 = 0.5$$

$$l_1 = (0.1)[10^3] = 100$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_2 = (0.1)f(0.05, 10.25, 55) = (0.1)(55) = 5.5$$

$$l_2 = (0.1)[(10.25)^3] = 107.7$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.1)f(0.05, 12.75, 58.85) = (0.1)(58.85) = 5.885$$

$$l_3 = (0.1)(12.75)^3 = 207.27$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$k_4 = (0.1)f(0.1, 15.885, 212.27) = (0.1)(212.27) = 21.227$$

$$l_4 = (0.1)(15.885)^3 = 400.83$$

We have, $y(x_0+h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$\therefore \quad y(0.1) = 10 + \frac{1}{6}\left[0.5 + 2(5.5) + 2(5.885) + 21.227\right]$$

Thus $\mathbf{y(0.1) = 17.4162}$

---

19. *Given $y'' - xy' - y = 0$ with the initial conditions $y(0) = 1$, $y'(0) = 0$, compute $y(0.2)$ and $y'(0.2)$ using fourth order Runge-Kutta method.*

>> Putting $y' = z$ we obtain $y'' = \frac{dz}{dx}$. The given equation becomes

$$\frac{dz}{dx} = xz + y \quad ; \quad y(0) = 1,\ z(0) = 0$$

Hence we have a system of equations,

$$\frac{dy}{dx} = z, \quad \frac{dz}{dx} = xz + y \quad \text{where } y = 1,\ z = 0,\ x = 0$$

Let $f(x, y, z) = z$, $g(x, y, z) = xz + y$, $x_0 = 0$, $y_0 = 1$, $z_0 = 0$ and $h = 0.2$

We shall first compute the following.

$$k_1 = hf(x_0, y_0, z_0) = (0.2)f(0, 1, 0) = (0.2)0 = 0$$

$$l_1 = (0.2)[0 \times 0 + 1] = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2},\ z_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.2)f(0.1, 1, 0.1) = (0.2)(0.1) = 0.02$$

$$l_2 = (0.2)[0.1 \times 0.1 + 1] = 0.202$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ z_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.2)f(0.1, 1.01, 0.101) = (0.2)(0.101) = 0.0202$$

$$l_3 = (0.2)[0.1 \times 0.101 + 1.01] = 0.204$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3,\ z_0 + l_3)$$

$$k_4 = (0.2)f(0.2, 1.0202, 0.204) = (0.2)(0.204) = 0.0408$$

$$l_4 = (0.2)[0.2 \times 0.204 + 1.0202] = 0.2122$$

We have $y(x_0+h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$z(x_0+h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

Substituting the approp .ate values we obtain $y(0.2) = 1.0202$ and $z(0.2) = 0.204$

Thus **$y(0.2) = 1.0202$ and $y'(0.2) = 0.204$**

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20. *Obtain the value of x and $\frac{dx}{dt}$ when $t = 0.1$ given that x satisfies the equation $\frac{d^2x}{dt^2} = t\frac{dx}{dt} - 4x$ and $x = 3$, $\frac{dx}{dt} = 0$ when $t = 0$ initially. Use fourth order Runge-Kutta method.*

>> Putting $y = \frac{dx}{dt}$ we obtain $\frac{dy}{dt} = \frac{d^2x}{dt^2}$ The given equation becomes

$$\frac{dy}{dt} = ty - 4x, \quad x = 3, \quad y = 0 \text{ when } t = 0$$

Hence we have a system of equations,

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = ty - 4x; \quad x = 3, \quad y = 0 \quad \text{when } t = 0.$$

Let $f(t, x, y) = y$, $g(t, x, y) = ty - 4x$, $t_0 = 0$, $x_0 = 3$, $y_0 = 0$ and $h = 0.1$

We shall first compute the following.

$$k_1 = hf(t_0, x_0, y_0) = (0.1)f(0, 3, 0) = (0.1)(0) = 0$$

$$l_1 = (0.1)[0 - 12] = -1.2$$

$$k_2 = hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}, y_0 + \frac{l_1}{2}\right)$$

$$k_2 = (0.1)f(0.05, 3, -0.6) = (0.1)(-0.6) = -0.06$$

$$l_2 = (0.1)[(0.05)(-0.6) - 12] = -1.203$$

$$k_3 = hf\left(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}, y_0 + \frac{l_2}{2}\right)$$

$$k_3 = (0.1)f(0.05, 2.97, -0.6015) = (0.1)(-0.6015) = -0.06015$$

$$l_3 = (0.1)[(0.05)(-0.6015) - 4 \times 2.97] = -1.191$$

$$k_4 = hf(t_0+h,\ x_0+k_3,\ y_0+l_3)$$

$$k_4 = (0.1)f(0.1,\ 2.93985,\ -1.191) = (0.1)(-1.191) = -0.1191$$

$$l_4 = (0.1)[(0.1)(-1.191) - 4\times 2.93985] = -1.18785$$

We have $x(t_0+h) = x_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$$y(t_0+h) = y_0 + \frac{1}{6}(l_1+2l_2+2l_3+l_4) \quad \text{where} \quad y = \frac{dx}{dt}$$

Substituting the appropriate values we obtain $x(0.1) = 2.9401$, $y(0.1) = -1.196$

Thus **$x = 2.9401$ and $\frac{dx}{dt} = -1.196$ when $t = 0.1$**

2.4 Milne's Method

Preamble : We recall [*Unit-I, Article-1.26*] Milne's predictor and corrector formulae for solving first order ODE : $y' = f(x, y)$; $y(x_0) = y_0$, $y(x_1) = y_1$, $y(x_2) = y_2$, $y(x_3) = y_3$. Here x_0, x_1, x_2, x_3 are equidistant values of x distant h.

We have to compute $y(x_4)$ where $x_4 = x_0 + 4h$

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') \qquad \text{[Predictor formula]}$$

$$y_4^{(c)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') \qquad \text{[Corrector formula]}$$

Method to solve $y'' = f(x, y, y')$ given $y(x_0) = y_0$ and $y'(x_0) = y_0'$

- We put $y' = z$ which gives $y'' = \frac{dz}{dx} = z'$.
  The given d.e becomes $z' = f(x, y, z)$
- We equip with the following table of values.

x	x_0	x_1	x_2	x_3
y	y_0	y_1	y_2	y_3
$y' = z$	$y_0' = z_0$	$y_1' = z_1$	$y_2' = z_2$	$y_3' = z_3$
$y'' = z'$	$y_0'' = z_0'$	$y_1'' = z_2'$	$y_2'' = z_2'$	$y_3'' = z_3'$

➲ We first apply predictor formula to compute $y_4^{(P)}$ and $z_4^{(P)}$ where,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3), \text{ since } y' = z.$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

➲ We compute $z_4' = f(x_4, y_4, z_4)$ and then apply corrector formula where,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

➲ Corrector formula can be applied repeatedly for better accuracy.

WORKED PROBLEMS

21. *Apply Milne's method to solve* $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$ *given* $y(0) = 1 = y'(0)$.
Compute $y(0.4)$ *by generating the initial values from Picard's method.*
Also compute $y(0.4)$ *theoretically.*

>> Putting $y' = \frac{dy}{dx} = z$, we obtain $y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx}$

The given equation becomes $\frac{dz}{dx} = 1 + z$ and now we have the system of equations,

$\frac{dy}{dx} = z : y = 1, x = 0$; $\frac{dz}{dx} = 1 + z$; $z = 1, x = 0$

or $dy = z\,dx$; $dz = (1 + z)\,dx$

Integrating LHS between 1 and y, RHS between 0 and x we have, ; Integrating LHS between 1 and z, RHS between 0 and x we have,

$\int_1^y dy = \int_0^x z\,dx$; $\int_1^z dz = \int_0^x (1 + z)\,dx$

Hence, $y = 1 + \int_0^x z\,dx$; $z = 1 + \int_0^x (1 + z)\,dx$

Now, $y_1 = 1 + \int_0^x 1\, dx \quad ; \quad z_1 = 1 + \int_0^x (1+1)\, dx$

$\therefore \quad y_1 = 1 + x \quad ; \quad z_1 = 1 + 2x$

$y_2 = 1 + \int_0^x z_1\, dx \quad ; \quad z_2 = 1 + \int_0^x [1 + z_1]\, dx$

$y_2 = 1 + \int_0^x (1+2x)\, dx \quad ; \quad z_2 = 1 + \int_0^x (2+2x)\, dx$

$\therefore \quad y_2 = 1 + x + x^2 \quad ; \quad z_2 = 1 + 2x + x^2$

$y_3 = 1 + \int_0^x z_2\, dx \quad ; \quad z_3 = 1 + \int_0^x [1 + z_2]\, dx$

$y_3 = 1 + \int_0^x (1 + 2x + x^2)\, dx \quad ; \quad z_3 = 1 + \int_0^x (2 + 2x + x^2)\, dx$

$\therefore \quad y_3 = 1 + x + x^2 + \frac{x^3}{3} \quad ; \quad z_3 = 1 + 2x + x^2 + \frac{x^3}{3}$

From y_3 and z_3 by putting $x = 0.1, 0.2, 0.3$ we obtain the following values

$y(0.1) = 1.1103, \ y(0.2) = 1.2427, \ y(0.3) = 1.399$

$z(0.1) = 1.2103, \ z(0.2) = 1.4427, \ z(0.3) = 1.699$

Further we have $z' = 1 + z$ and hence,

$z'(0) = 1 + z(0) = 1 + 1 = 2$

$\therefore \quad z'(0.1) = 1 + z(0.1) = 2.2103$

$z'(0.2) = 1 + z(0.2) = 2.4427$

$z'(0.3) = 1 + z(0.3) = 2.699$

Now we tabulate these values.

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$
y	$y_0 = 1$	$y_1 = 1.1103$	$y_2 = 1.2427$	$y_3 = 1.399$
$y' = z$	$z_0 = 1$	$z_1 = 1.2103$	$z_2 = 1.4427$	$z_3 = 1.699$
$y'' = z'$	$z_0' = 2$	$z_1' = 2.2103$	$z_2' = 2.4427$	$z_3' = 2.699$

We first consider Milne's predictor formulae :

$$y_4^{(P)} = y_0 + \frac{4h}{3}\,(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\,(2z_1' - z_2' + 2z_3')$$

Hence, $y_4^{(P)} = 1 + \frac{4(0.1)}{3}\,[2(1.2103) - 1.4427 + 2(1.699)]$

$$z_4^{(P)} = 1 + \frac{4(0.1)}{3}\,[2(2.2103) - 2.4427 + 2(2.699)]$$

$\therefore \quad y_4^{(P)} = 1.5835$ and $z_4^{(P)} = 1.9835$

Next we consider Milne's corrector formulae :

$$y_4^{(C)} = y_2 + \frac{h}{3}\,(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}\,(z_2' + 4z_3' + z_4')$$

We have, $z_4' = 1 + z_4^{(P)} = 1 + 1.9835 = 2.9835$

Hence, $y_4^{(C)} = 1.2427 + \frac{0.1}{3}\,[1.4427 + 4(1.699) + 1.9835]$

$$z_4^{(C)} = 1.4427 + \frac{0.1}{3}\,[2.4427 + 4(2.699) + 2.9835]$$

$\therefore \quad y_4^{(C)} = 1.58344$ and $z_4^{(C)} = 1.98344$

Applying the corrector formula again for y_4 we obtain $y_4^{(C)} = 1.583438$

Thus the required $\mathbf{y(0.4) = 1.5834}$

Theoretical solution of the problem is as follows.

$$y'' = 1 + y \quad \text{or} \quad y'' - y' = 1 \quad \text{or} \quad (D^2 - D)\,y = 1$$

A.E : $m^2 - m = 0$ or $m(m-1) = 0 \Rightarrow m = 0, 1$

$$y_c = c_1 + c_2 e^x$$

$$y_p = \frac{1}{D^2 - D} = \frac{e^{0x}}{D^2 - D} = x \cdot \frac{e^{0x}}{2D - 1} = -x$$

Complete solution : $y = y_c + y_p$ is given by $y = c_1 + c_2 e^x - x$

This gives $y' = c_2 e^x - 1$ and by using $y(0) = 1$, $y'(0) = 1$ we have,

$$1 = c_1 + c_2 \text{ and } 1 = c_2 - 1 \quad \therefore \quad c_2 = 2 \text{ and } c_1 = -1$$

Hence, $y = -1 + 2e^x - x$ is the theoretical solution.

Now $y(0.4) = -1 + 2e^{0.4} - 0.4 = 1.58365$, **theoretically.**

---

22. *Apply Milne's method to compute $y(0.8)$ given that $\dfrac{d^2y}{dx^2} = 1 - 2y\dfrac{dy}{dx}$ and the following table of initial values.*

x	0	0.2	0.4	0.6
y	0	0.02	0.0795	0.1762
y'	0	0.1996	0.3937	0.5689

Apply the corrector formula twice in presenting the value of y at $x = 0.8$

>> Putting $y' = \dfrac{dy}{dx} = z$ we obtain $y'' = \dfrac{d^2y}{dx^2} = z'$

The given equation becomes $z' = 1 - 2yy' = 1 - 2yz$

Now, $z_0' = 1 - 2(0)(0) = 1$

$$z_1' = 1 - 2(0.02)(0.1996) = 0.992$$

$$z_2' = 1 - 2(0.0795)(0.3937) = 0.9374$$

$$z_3' = 1 - 2(0.1762)(0.5689) = 0.7995$$

We have the following table.

x	$x_0 = 0$	$x_1 = 0.2$	$x_2 = 0.4$	$x_3 = 0.6$
y	$y_0 = 0$	$y_1 = 0.02$	$y_2 = 0.0795$	$y_3 = 0.1762$
$y' = z$	$z_0 = 0$	$z_1 = 0.1996$	$z_2 = 0.3937$	$z_3 = 0.5689$
$y'' = z'$	$z_0' = 1$	$z_1' = 0.992$	$z_2' = 0.9374$	$z_3' = 0.7995$

We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

On substituting the appropriate values from the table we obtain

$$y_4^{(P)} = 0.3049 \quad \text{and} \quad z_4^{(P)} = 0.7055$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

We have $z_4' = 1 - 2y_4^{(P)} z_4^{(P)} = 1 - 2(0.3049)(0.7055) = 0.5698$

Hence by substituting the appropriate values in the corrector formulae we obtain

$$y_4^{(C)} = 0.3045 \quad \text{and} \quad z_4^{(C)} = 0.7074$$

Applying the corrector formula again for y_4 we have

$$y_4^{(C)} = 0.0795 + \frac{0.2}{3}[0.3937 + 4(0.5689) + 0.7074] = 0.3046$$

Thus the required $y(0.8) = \mathbf{0.3046}$

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23. *Obtain the solution of the equation* $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ *by computing the value of the dependent variable corresponding to the value 1.4 of the independent variable by applying Milne's method using the following data*

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
y'	2	2.3178	2.6725	3.0657

>> Dividing the given equation by 2 we have,

$$\frac{d^2y}{dx^2} = 2x + \frac{1}{2}\frac{dy}{dx} \quad \text{or} \quad y'' = 2x + \frac{y'}{2}$$

Putting $y' = z$ we obtain $y'' = z'$ and the given equation becomes $z' = 2x + \frac{z}{2}$

Now, $z_0' = 2(1) + \frac{2}{2} = 3$

$$z_1' = 2(1.1) + \frac{2.3178}{2} = 3.3589$$

$$z_2' = 2(1.2) + \frac{2.6725}{2} = 3.73625$$

$$z_3' = 2(1.3) + \frac{3.0657}{2} = 4.13285$$

We have the following table.

x	$x_0 = 1$	$x_1 = 1.1$	$x_2 = 1.2$	$x_3 = 1.3$
y	$y_0 = 2$	$y_1 = 2.2156$	$y_2 = 2.4649$	$y_3 = 2.7514$
$y' = z$	$z_0 = 2$	$z_1 = 2.3178$	$z_2 = 2.6725$	$z_3 = 3.0657$
$y'' = z'$	$z_0' = 3$	$z_1' = 3.3589$	$z_2' = 3.73625$	$z_3' = 4.13285$

We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}\,(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\,(2z_1' - z_2' + 2z_3')$$

On substituting the appropriate values from the table we obtain

$$y_4^{(P)} = 3.0793 \quad \text{and} \quad z_4^{(P)} = 3.4996$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}\,(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}\,(z_2' + 4z_3' + z_4')$$

We have, $z_4' = 2x_4 + \dfrac{z_4^{(P)}}{2} = 2(1.4) + \dfrac{3.4996}{2} = 4.5498$

Hence by substituting the appropriate values in the corrector formulae we obtain

$$y_4^{(C)} = 3.0794 \quad \text{and} \quad z_4^{(C)} = 3.4997$$

Thus the required value of y is 3.0794 at $x = 1.4$.

---

24. *Apply Picard's method to find the third approximation to the solution of the equation $y'' + xy' + y = 0$; $y(0) = 1$, $y'(0) = 0$. Then apply Milne's method to compute $y(0.4)$ by generating the requisite initial values from Picard's approximations.*

>> Putting $y' = z$ we obtain $y'' = \dfrac{dz}{dx} = z'$

The given equation becomes $z' + xz + y = 0$ and now we have a system of equations

$$\frac{dy}{dx} = z : x = 0, y = 1 \quad ; \quad \frac{dz}{dx} = -(xz+y) ; x = 0, z = 0$$

or $dy = z\,dx$; $dz = -(xz+y)\,dx$

Integrating LHS between 1 and y, RHS between 0 and x we have, ; Integrating LHS between 0 and z, RHS between 0 and x we have,

$$\int_1^y dy = \int_0^x z\,dx \quad ; \quad \int_0^z dz = -\int_0^x (xz+y)\,dx$$

Hence, $y = 1 + \int_0^x z\,dx$; $z = -\int_0^x (xz+y)\,dx$

Now, $y_1 = 1 + \int_0^x 0\,dx = 1$; $z_1 = -\int_0^x 1\,dx = -x$

$$y_2 = 1 + \int_0^x z_1\,dx \quad ; \quad z_2 = -\int_0^x [x z_1 + y_1]\,dx$$

$$y_2 = 1 + \int_0^x -x\,dx \quad ; \quad z_2 = -\int_0^x [x(-x)+1]\,dx$$

$$\therefore \quad y_2 = 1 - \frac{x^2}{2} \quad ; \quad z_2 = -x + \frac{x^3}{3}$$

$$y_3 = 1 + \int_0^x z_2\,dx \quad ; \quad z_3 = -\int_0^x [x z_2 + y_2]\,dx$$

$$y_3 = 1 + \int_0^x \left(-x + \frac{x^3}{3}\right) dx \quad ; \quad z_3 = -\int_0^x \left(-x^2 + \frac{x^4}{3} + 1 - \frac{x^2}{2}\right) dx$$

$$\therefore \quad y_3 = 1 - \frac{x^2}{2} + \frac{x^4}{12} \quad ; \quad z_3 = -x + \frac{x^3}{2} - \frac{x^5}{15}$$

From y_3 and z_3 by putting $x = 0.1, 0.2, 0.3$ we obtain

$$y(0.1) = 0.995, \quad y(0.2) = 0.9801, \quad y(0.3) = 0.956$$

$$z(0.1) = -0.0995, \quad z(0.2) = -0.196, \quad z(0.3) = -0.2867$$

Also we have $z' = -(xz+y)$ and we have

$$z'(0) = -[0+1] = -1$$

$$z'(0.1) = -[(0.1)(-0.0995)+0.995] = -0.985$$

$$z'(0.2) = -[(0.2)(-0.196)+0.9801] = -0.941$$

$$z'(0.3) = -[(0.3)(-0.2867)+0.956] = -0.87$$

Now we tabulate all these values.

x	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$	$x_3 = 0.3$
y	$y_0 = 1$	$y_1 = 0.995$	$y_2 = 0.9801$	$y_3 = 0.956$
$y' = z$	$z_0 = 0$	$z_1 = -0.0995$	$z_2 = -0.196$	$z_3 = -0.2867$
$y'' = z'$	$z_0' = -1$	$z_1' = -0.985$	$z_2' = -0.941$	$z_3' = -0.87$

We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

On substituting the appropriate values from the table we obtain

$$y_4^{(P)} = 0.9231 \quad \text{and} \quad z_4^{(P)} = -0.3692$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

We have $z_4' = -(x_4 z_4^{(P)} + y_4^{(P)}) = -[(0.4)(-0.3692)+0.9231] = -0.7754$

Hence by substituting the appropriate values in the corrector formulae we obtain

$$y_4^{(C)} = 0.9230 \quad \text{and} \quad z_4^{(C)} = -0.3692$$

Thus the required $\mathbf{y(0.1) = 0.923}$

---

25. *Applying Milne's predictor and corrector formulae compute y (0.8) given that y satisfies the equation $y'' = 2yy'$ and y & y' are governed by the following values*

$$y(0) = 0, \quad y(0.2) = 0.2027, \quad y(0.4) = 0.4228, \quad y(0.6) = 0.6841$$

$$y'(0) = 1, \quad y'(0.2) = 1.041, \quad y'(0.4) = 1.179, \quad y'(0.6) = 1.468$$

Apply corrector formula twice.

>> Putting $y' = z$ we obtain $y'' = \frac{dz}{dx} = z'$ & the given equation becomes $z' = 2yz$

Now, $z'(0) = 0, \quad z'(0.2) = 2(0.2027)(1.041) = 0.422$

$$z'(0.4) = 2(0.4228)(1.179) = 0.997$$

$$z'(0.6) = 2(0.6841)(1.468) = 2.009$$

Now we tabulate all the values.

x	$x_0 = 0$	$x_1 = 0.2$	$x_2 = 0.4$	$x_3 = 0.6$
y	$y_0 = 0$	$y_1 = 0.2027$	$y_2 = 0.4228$	$y_3 = 0.6841$
$y' = z$	$z_0 = 1$	$z_1 = 1.041$	$z_2 = 1.179$	$z_3 = 1.468$
$y'' = z'$	$z_0' = 0$	$z_1' = 0.422$	$z_2' = 0.997$	$z_3' = 2.009$

We first consider Milne's predictor formulae,

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

On substituting the appropriate values from the table we obtain

$$y_4^{(P)} = 1.0237 \quad \text{and} \quad z_4^{(P)} = 2.0307$$

Next we consider Milne's corrector formulae,

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

We have $z_4' = 2y_4^{(P)} z_4^{(P)} = 4.1577$

Hence by substituting the appropriate values in the corrector formulae we obtain

$$y_4^{(C)} = 1.0282 \quad \text{and} \quad z_4^{(C)} = 2.0584$$

Applying the corrector formula again we have

$$y_4^{(C)} = 0.4228 + \frac{0.2}{3}\,[\,1.179 + 4\,(\,1.468\,) + 2.0584\,] = 1.03009$$

Thus the required $y\,(\,0.8\,) = 1.0301$

EXERCISES

1. Use Picard's second approximation to compute $y\,(\,0.1\,)$ and $z\,(\,0.1\,)$ by solving the system of equations

   $$\frac{dy}{dx} = x + z\,,\ \frac{dz}{dx} = x - y^2\ ;\ y\,(\,0\,) = 2, z\,(\,0\,) = 1$$

2. Use Picard's method to solve the system of equations

   $\frac{dy}{dx} = y + 2z,\ \frac{dz}{dx} = 3y + 2z\ ;\ y = 6,\ z = 4$ when $x = 0$ by finding three approximations. Hence find y and z at $x = 0.02$

3. Use the fourth order Runge-Kutta method to obtain the solution of the following system of equations at $x = 0.3$

   $$\frac{dy}{dx} = 1 + xz\,,\ \frac{dz}{dx} + xy = 0\,;\ x = 0\,,\ y = 0\,,\ z = 1$$

4. Solve the following system of equation using Runge- Kutta method of order 4 by taking $h = 0.1$

   $$\frac{dx}{dt} = y - t\,,\ \frac{dy}{dx} = x + t\ ;\ x = 1\,,\ y = 1 \text{ at } t = 0$$

5. Use Picard's method to solve $\frac{d^2y}{dx^2} = x\frac{dy}{dx} + y$ given $y\,(\,1\,) = 2 = y'\,(\,1\,)$.

   Compute $y\,(\,1.2\,)$ correct to two decimal places.

6. Solve $y'' = 1 + 2\,y\,y'\,;\ y\,(\,0\,) = 1\,,\ y'\,(\,0\,) = 1$ by applying Picards method. Compute $y\,(\,0.2\,)$

7. Use fourth order Runge-Kutta method to solve the equation $\frac{d^2y}{dx^2} = x\frac{dy}{dx} + y$ given that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$. Compute y and $\frac{dy}{dx}$ at $x = 0.2$

8. Solve $y'' + 4y = x\,y$ given that $y\,(\,0\,) = 3$ and $y'\,(\,0\,) = 0$. Compute $y\,(\,0.1\,)$ using Runge-Kutta method of order 4.

9. Apply Milne's method to compute $y\,(\,0.4\,)$ given the equation $y'' + y' = 2\,e^x$ and the following table of initial values. Compare the result with the theoretical value.

x	0	0.1	0.2	0.3
y	2	2.01	2.04	2.09
y'	0	0.2	0.4	0.6

10. Solve the equation $y'' + y' = 2x$ at $x = 0.4$ by applying Milne's method given that $y = 1$, $y' = -1$ at $x = 0$. Requisite initial values be generated from Picard's method.

ANSWERS

1. $y(0.1) = 2.0845, z(0.1) = 0.5867$
2. $y_3 = 6 + 14x + 33x^2 + \frac{127}{3}x^3$; $z_3 = 4 + 26x + 47x^2 + \frac{193}{3}x^3$;

   $y(0.02) = 6.2935,\ z(0.02) = 4.5393$
3. $y(0.3) = 0.3448,\ z(0.3) = 0.9899$
4. $x(0.1) = 1.1003,\ y(0.1) = 1.1102$
5. $y(1.2) = 2.49$
6. $y(0.2) = 1.2736$
7. $y(0.2) = 1.0202,\ y'(0.2) = 0.204$
8. $y(0.1) = 2.94$
9. $y(0.4) = 2.16$
10. $y(0.4) = 0.6897$